INDETERMINATE FORMS

If $f(x)$ and $g(x)$ are two functions, then we know that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then the expression $\frac{f(x)}{g(x)}$ is said to have the indeterminat form $\frac{0}{0}$, at $x = 0$.

If $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} g(x) = \infty$, then $\frac{f(x)}{g(x)}$ is said to have the indeterminate form $\frac{\infty}{\infty}$.

The other indeterminate forms are $\infty - \infty$, $0 \times \infty$, 0^0, 1^∞, ∞^0.

2.1.1 Indeterminate Form $\frac{0}{0}$

Here, we shall give a method called L' Hospital's rule to evaluate the limits of the expressions which take the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

L' Hospital's Theorem

Let $f(x)$ and $g(x)$ be two functions such that

(1) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

(2) $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$

then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

If $f'(a) = 0 = g'(a)$, then this theorem can be extended as follows:

$$\lim_{c \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

$$= \lim_{x \to a} \frac{f'''(x)}{g'''(x)}, \text{ if } f''(a) = 0 = g''(a)$$

and so on.

WORKED OUT EXAMPLES

1. *Evaluate*: $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution $L = \lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\left[\frac{0}{0}\right]$ form

By L' Hospital rule

$$L = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

2. *Evaluate*: $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$.

Solution $L = \lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$ $\left[\frac{0}{0}\right]$ form

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x \cos x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x \to 0} \frac{\sin x}{-\sin^3 x + 2 \sin x \cos^2 x}$$

$$= \lim_{x \to 0} \frac{1}{-\sin^2 x + 2 \cos^2 x} = \frac{1}{-0 + 2} = \frac{1}{2}.$$

3. *Evaluate*: $\lim_{x \to 0} \frac{a^x - b^x}{\sin x}$.

Solution $L = \lim_{x \to 0} \frac{a^x - b^x}{\sin x}$ $\left[\frac{0}{0}\right]$ form

By L' Hospital rule

$$= \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{\cos x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \frac{\log a - \log b}{1}$$

$$= \log\left(\frac{a}{b}\right).$$

4. *Evaluate*: $\lim_{x \to 0} \frac{x \sin x}{\left(e^x - 1\right)^2}$.

Solution $L = \lim_{x \to 0} \frac{x \sin x}{\left(e^x - 1\right)^2}$ $\left[\frac{0}{0}\right]$ form

By L' Hospital rule

$$L = \lim_{x \to 0} \frac{x \cos x + \sin x}{2\left(e^x - 1\right) \cdot e^x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{-x \sin x + \cos x + \cos x}{2\left[2e^{2x} - e^x\right]}$$

$$= \frac{2}{2\,(2-1)} = 1.$$

10. *Evaluate*: $\lim_{x \to 0} \frac{e^{2x} - (1+x)^2}{x \log(1+x)}$.

Solution $L = \lim_{x \to 0} \frac{e^{2x} - (1+x)^2}{x \log(1+x)}$ $\left[\frac{0}{0}\right]$ form

By L' Hospital rule

$$= \lim_{x \to 0} \frac{2e^{2x} - 2(1+x)}{\frac{x}{1+x} + \log(1+x)} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{4e^{2x} - 2}{\frac{1}{(1+x)^2} + \frac{1}{1+x}} = \frac{4-2}{1+1} = 1.$$

11. *Evaluate: If* $\lim_{x \to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3} = \frac{1}{3}$*, find a and b.*

Solution

Given $\frac{1}{3} = \lim_{x \to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3} \qquad \left[\frac{0}{0}\right]$ form

By L' Hospital rule

$$= \lim_{x \to 0} \frac{1 - a\cos x + x\,a\sin x + b\cos x}{3x^2}$$

At $x = 0$, the numerator $= 1 - a + b$.

In order that the limit should exist, we must have

$$1 - a + b = 0 \qquad ...(1)$$

Applying L' Hospital rule with this assumption, we get

$$\frac{1}{3} = \lim_{x \to 0} \frac{a\sin x + a(x\cos x + \sin x) - b\sin x}{6x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{a\cos x + a(-x\sin x + \cos x + \cos x) - b\cos x}{6}$$

$$= \frac{a + 2a - b}{6}$$

$$\frac{1}{3} = \frac{3a - b}{6}$$

Hence, $3a - b = 2$ ...(2)

Solving (1) and (2), we get $a = \frac{1}{2}$, $b = \frac{-1}{2}$.

2.1.2 Indeterminate Forms $\infty - \infty$ and $0 \times \infty$

L' Hospital rule can be applied to limits which take the indeterminate forms $\infty - \infty$ and $0 \times \infty$. First we transform the given limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then by use L'Hospital rule to evaluate the limit.

WORKED OUT EXAMPLES

1. *Evaluate:* $\lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2} \right]$.

Solution $L = \lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2} \right]$ $[\infty - \infty]$ form

Hence, required limit $L = \lim_{x \to 0} \frac{x - \log(1+x)}{x^2}$ $\left[\frac{0}{0}\right]$ form

By L' Hospital rule $= \lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2x}$

$$= \lim_{x \to 0} \frac{\frac{x}{1+x}}{2x}$$

$$= \lim_{x \to 0} \frac{1}{2(1+x)}$$

$$= \frac{1}{2}.$$

4. *Evaluate:* $\lim_{x \to 0} \tan x \log x$.

Solution $L = \lim_{x \to 0} \tan x \log x$ $[0 \times (-\infty)]$ form

$= \lim_{x \to 0} \frac{\log x}{\cot x}$ $\left[-\frac{\infty}{\infty}\right]$ form

By L' Hospital rule $= \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{-2\sin x \cos x}{1} = 0.$$

5. *Evaluate:* $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$.

Solution $$L = \lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \qquad [\infty - \infty] \text{ form}$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \times \frac{x^2}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \times \lim_{x\to 0} \frac{x^2}{\sin^2 x}$$

$$\left(\lim_{x\to 0} \frac{\sin x}{x} = 1 \therefore \lim_{x\to 0} \frac{x}{\sin x} = 1\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By applying L' Hospital rule

$$= \lim_{x\to 0} \frac{2\sin x \cos x - 2x}{4x^3}$$

$$= \lim_{x\to 0} \frac{\sin 2x - 2x}{4x^3} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{12x^2}$$

$$= \lim_{x\to 0} \frac{\cos 2x - 1}{6x^2} \qquad (\because 1 - \cos 2x = 2\sin^2 x)$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{6x^2}$$

$$= \lim_{x\to 0} -\frac{1}{3}\left(\frac{\sin x}{x}\right)^2$$

$$= \frac{-1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$= \frac{-1}{3}\times 1$$

$$= \frac{-1}{3}.$$

7. *Evaluate*: $\lim_{x\to\frac{\pi}{4}} (1 - \tan x) \sec 2x.$

Solution $L = \lim_{x\to\frac{\pi}{4}} (1 - \tan x) \sec 2x$ $[0 \times \infty]$ form

$$= \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\cos 2x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1.$$

8. *Evaluate*: $\lim_{x\to a} \log\left(2-\frac{x}{a}\right)\cot(x-a).$

Solution $L = \lim_{x\to a} \log\left(2-\frac{x}{a}\right)\cot(x-a)$ $[0 \times \infty]$ form

$$= \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\tan(x-a)} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule
$$= \lim_{x\to a} \frac{\frac{1}{\left(2-\frac{x}{a}\right)} \times \left(\frac{-1}{a}\right)}{\sec^2(x-a)}$$

$$= \frac{\left(-\frac{1}{a}\right)}{1} = \frac{-1}{a}.$$

2.1.3 Indeterminate Forms 0^0, 1^∞, ∞^0, 0^∞

Let $L = \lim_{x\to a} \{f(x)\}^{g(x)}$. If L takes one of the indeterminate forms 0^0, 1^∞, ∞^0, 0^0 then taking logarithm on both sides, we get

$$\log L = \lim_{x\to a} g(x) \log f(x)$$

Then, $\log L$ will take the indeterminate form $0 \times \infty$ and which can be evaluated by using the method employed in preceding section.

WORKED OUT EXAMPLES

1. *Evaluate*: $\lim_{x\to 0} (1 + \sin x)^{\cot x}$.

Solution $L = \lim_{x\to 0} (1 + \sin x)^{\cot x}$ $[1^\infty]$ form

Taking logarithm on both sides

$$\log L = \lim_{x\to 0} \cot x \log(1 + \sin x) \quad [\infty \times 0] \text{ form}$$

$$= \lim_{x\to 0} \frac{\log(1+\sin x)}{\tan x} \quad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x\to 0} \frac{\cos x / 1 + \sin x}{\sec^2 x}$$

$$\log L = \lim_{x\to 0} \frac{\cos x}{\sec^2 x(1+\sin x)} = \frac{1}{1(1+0)} = 1$$

$\therefore$ $L = e^1 = e.$

2. *Evaluate*: $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$.

Solution $$L = \lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x} \qquad (1^{\infty}) \text{ form}$$

Taking logarithm on both sides

$$\log L = \lim_{x \to \frac{\pi}{4}} \tan 2x \log (\tan x) \qquad [\infty \times 0] \text{ form}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\log \tan x}{\cot 2x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x / \tan x}{-2 \operatorname{cosec}^2 2x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{2 \tan x \operatorname{cosec}^2 2x}$$

$$= \frac{-\left(\sqrt{2}\right)^2}{2 \cdot 1 \cdot 1^2}$$

$$\log L = -1$$

$$L = e^{-1}$$

$$= \frac{1}{e}.$$

3. *Evaluate*: $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

Solution $$L = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \qquad [1^{\infty}] \text{ form}$$

(Since, $\lim_{x \to 0} \frac{\tan x}{x} = 1$ as $x \to 0$)

Taking logarithm on both sides

$$\log L = \lim_{x \to 0} \frac{1}{x^2} \log \frac{\tan x}{x} \qquad [\infty \times 0] \text{ form}$$

$$= \lim_{x \to 0} \frac{\log \frac{\tan x}{x}}{x^2} \qquad \left[\frac{0}{0}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x \to 0} \frac{\frac{1}{\frac{\tan x}{x}} \cdot \left[\frac{x \sec^2 x - \tan x}{x^2}\right]}{2x}$$

$$= \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{2\left[x^2 \sec^2 x + 2x \tan x\right]}$$

$$= \lim_{x \to 0} \frac{\sec^2 x \tan x}{x \sec^2 x + 2 \tan x} \qquad \left[\frac{0}{0}\right] \text{ form}$$

Again By L' Hospital Rule

$$\log L = \lim_{x \to 0} \frac{\sec^2 x \sec^2 x + \tan x \cdot 2 \sec^2 x \tan x}{\sec^2 x + x\left(2 \sec^2 x \tan x\right) + 2 \sec^2 x} = \frac{1}{3}$$

$$\therefore \qquad L = e^{1/3}.$$

4. *Evaluate*: $\lim_{x \to \infty} (1 + x^2)^{e^{-x}}$.

Solution $$L = \lim_{x \to \infty} (1 + x^2)^{e^{-x}} \qquad \left(\begin{array}{l} \therefore e^{-\infty} = 0, \\ 1 + \infty = \infty \end{array}\right) \quad [\infty^0] \text{ form}$$

Taking log on both sides

$$\log L = \lim_{x \to \infty} e^{-x} \log (1 + x^2) \qquad [0 \times \infty] \text{ form}$$

$$= \lim_{x \to \infty} \frac{\log\left(1 + x^2\right)}{e^x} \qquad \left[\frac{\infty}{\infty}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x \to \infty} \frac{2x/(1+x^2)}{e^x}$$

$$= \lim_{x \to \infty} \frac{2x}{(1+x^2)\, e^x} \qquad \left[\frac{\infty}{\infty}\right] \text{ form}$$

$$= \lim_{x \to \infty} \frac{2}{(1+x^2)\, e^x + 2x\, e^x}$$

$$\log L = 0$$

$$L = e^0 = 1.$$

5. *Evaluate*: $\lim\limits_{x \to \frac{\pi}{2}} (\sec x)^{\cot x}$.

Solution

$$L = \lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x} \qquad [\infty^0] \text{ form}$$

$$\log L = \lim_{x \to \frac{\pi}{2}} \cot x \log \sec x \qquad [0 \times \infty] \text{ form}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log \sec x}{\tan x} \qquad \left[\frac{\infty}{\infty}\right] \text{ form}$$

By L' Hospital rule

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\sec x \cdot \tan x}{\sec x}}{\sec^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec^2 x}$$

$$\log L = \lim_{x \to \frac{\pi}{2}} \sin x \cos x = 0$$

$$\therefore \quad L = e^0 = 1$$