APPENDIX—A

MISCELLANEOUS PROBLEMS WITH SOLUTIONS

❖ **Example 1.**

Optimize the utility function $U = 4xy - y^2$ subject to the budget constraint $2x + y = 6$.

○ **Solution :**

Given the utility function : $U = f(x, y) = 4xy - y^2$ and the budget constraint $g(x, y) = 2x + y - 6 = 0$. Let us construct the Lagrangian expression : $F(x, y,) = 4xy - y^2 + \lambda(2x + y - 6)$ where λ is the Lagrange multiplier. The first order conditions for maximum utility are :

$$\frac{\delta F}{\delta x} = 4y + 2\lambda = 0 \qquad ...(1)$$

$$\frac{\delta F}{\delta y} = 4x - 2y + \lambda = 0 \qquad ...(2)$$

$$\frac{\delta F}{\delta \lambda} = 2x + y - 6 = 0 \qquad ...(3)$$

Solving these three equation for x, y and λ, we have

$$4y = -2\lambda \quad \text{or,} \quad y = \frac{-1}{2}\lambda$$

$$4x = 2y - \lambda \quad \text{or,} \quad 4x = 2\left(-\frac{1}{2}\lambda\right) - \lambda \quad \text{or,} \quad 4x = -2\lambda \quad \text{or,} \quad x = \frac{-1}{2}\lambda$$

$$\therefore 2\left(\frac{-1}{2}\lambda\right) + \left(\frac{-1}{2}\lambda\right) - 6 = 0 \quad \text{or,} \quad -\lambda - \frac{\lambda}{2} = 6 \quad \text{or,} \quad -3\lambda = 12 \quad \text{or,} \quad \lambda = -4$$

$\therefore x = y = 2$ and $\lambda = -4$

The second order condition for maximum utility requires :

$$\begin{vmatrix} 0 & 4 & 2 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{vmatrix} > 0 = -4(-2) + 2(4 + 4) = 24 > 0$$

Hence the utility function has a maximum value at (2, 2) and

$$U_{max} = 4 \times 2 \times 2 - (2)^2 = 12.$$

❖ **Example 2.**

Utility function for two goods is given by $U = (x + 2)(y + 1)$. It is given that p_x = Rs 4, p_y = Rs. 6 and the individual's fixed income is Rs 130. Using the Lagrange multiplier method, find the optimum levels of purchase of the two commodities. Is the second order condition for maximum utility satisfied ?

○ Solution :

Given the utility function $U = f(x, y) = (x + 2)(y + 1)$ ... (1) and budget constraint $g(x, y) = 4x + 6y - 130 = 0$ ... (2) consider the Lagrangian expression:

$$F(x, y, \lambda) = (x + 2)(y + 1) + \lambda(4x + 6y - 130) \quad ...(3)$$

The first order conditions for utility maximisation are

$$\frac{\delta F}{\delta x} = (y+1) + 4\lambda = 0 \quad ...(4)$$

$$\frac{\delta F}{\delta y} = (x+2) + 6\lambda = 0 \quad ...(5)$$

$$\frac{\delta F}{\delta \lambda} = 4x + 6y - 130 = 0 \quad ...(6)$$

From (4), $y = (-1 - 4\lambda)$ From (5), $x = (-2 - 6\lambda)$. Putting these values in (6) we have

$$4(-2 - 6\lambda) + 6(-1 - 4\lambda) - 130 = 0$$

or, $-8 - 24\lambda - 6 - 24\lambda - 130 = 0$

or, $-48\lambda = 144$ or, $\lambda = -3$.

Then $x = -2 - 6(-3) = 18 - 2 = 16$ and $y = -1 - 4(-3) = 12 - 1 = 11$

The second order condition for maximum utility requires

$$\begin{vmatrix} 0 & 1 & 4 \\ 1 & 0 & 6 \\ 4 & 6 & 0 \end{vmatrix} > 0 \text{ or, } -1(-24) + 4(6) \text{ or, } 24 + 24 = 48 > 0$$

$\therefore$ U has a maximum value at (16, 11) and $U_{max} = (16 + 2)(11 + 1) = 216$.

❖ Example 3.

A consumer has the following utility function derived over x_1 and x_2 : $U(x_1, x_2) = a_1 \log x_1 + a_2 \log x_2$; $a_1 + a_2 = 1$. Find his demand schedules for x_1 and x_2.

○ Solution :

Maximise $U(x_1, x_2) = a_1 \log x_1 + a_2 \log x_2$ ...(1)

subject to $\mu = x_1p_1 + x_2p_2$ ...(2)

where μ is money income of the consumer.

Consider the Lagrangian expression

$$F = a_1 \log x_1 + a_2 \log x_2 + \lambda(\mu - x_1p_1 - x_2p_2) \quad ...(3)$$

The first order conditions for maximum utility are :

$$\frac{\delta F}{\delta x_1} = \frac{a_1}{x_1} - \lambda p_1 = 0 \quad ...(4)$$

$$\frac{\delta F}{\delta x_2} = \frac{a_2}{x_2} - \lambda p_2 = 0 \quad ...(5)$$

$$\frac{\delta F}{\delta \lambda} = \mu - x_1p_1 - x_2p_2 = 0 \quad ...(6)$$

From (4) and (5) we get

$$\frac{a_1}{x_1} = \lambda p_1 \quad \text{or,} \quad \frac{a_1}{x_1 p_1} = \lambda \quad \text{and} \quad \frac{a_2}{x_2} = \lambda p_2 \quad \text{or,} \quad \frac{a_2}{x_2 p_2} = \lambda$$

$$\therefore \frac{a_1}{x_1 p_1} = \frac{a_2}{x_2 p_2} \quad \text{or, } x_1 p_1 . a_2 = x_2 p_2 . a_1 \quad \text{or, } x_1 p_1 = \frac{x_2 p_2 a_1}{a_2}$$

From (6) we have $x_1 p_1 + x_2 p_2 = \mu$. Then

$$\frac{x_2 p_2 . a_1}{a_2} + x_2 p_2 = \mu \qquad \text{or, } x_2 p_2\left(\frac{a_1}{a_2} + 1\right) = \mu$$

$$\text{or, } x_2 p_2\left(\frac{a_1 + a_2}{a_2}\right) = \mu \quad \text{or, } x_2 p_2\left(\frac{1}{a_2}\right) = \mu \quad \text{or, } x_2 = \frac{\mu a_2}{p_2}$$

Similarly it can be shown that $x_1 = \frac{\mu a_1}{p_1}$. These are the two required demand functions.

◆ Example 4.

Let the utility function be given by U = xy and the budget constraint be given as $100 - p_x x - p_y y = 0$. (i) Find the demand functions for x and y. (ii) Show that these functions are homogeneous of degree zero in absolute prices.

○ Solution :

(i) Utility function U = xy ... (1),

budget constraint $100 - p_x x - p_y y = 0$ ...(2)

Lagrangian expression $F = xy + \lambda(100 - p_x . x - p_y . y)$ ...(3)

First order conditions for maximum utility are :

$$\frac{\delta F}{\delta x} = y - \lambda p_x = 0 \qquad ...(4)$$

$$\frac{\delta F}{\delta y} = x - \lambda p_y = 0 \qquad ...(5)$$

$$\frac{\delta F}{\delta \lambda} = 100 - p_x . x - p_y . y = 0 \qquad ...(6)$$

From (4) $y = \lambda p_x$ or, $y/p_x = \lambda$

From (5) $x = \lambda p_y$ or, $\frac{x}{p_y} = \lambda$

$$\therefore \frac{y}{p_x} = \frac{x}{p_y} \quad \text{or, } x.p_x = y.p_y$$

From (6), we have

$$100 = xp_x + y.p_y \quad \text{or, } 100 = 2x.p_x \quad \text{or, } \frac{50}{p_x} = x$$

Again from (6) we have

$$100 = 2yp_y \quad \text{or, } \frac{50}{p_y} = y$$

∴ Demand functions for x and y are $x = \frac{50}{p_x}$ and $y = \frac{50}{p_y}$ respectively.

(ii) If p_x is replaced by kp_x and p_y is replaced by kp_y and 50 is replaced by $k.50$ then

$$x = \frac{k \cdot 50}{kp_x} = \frac{50}{p_x} \quad \text{and} \quad y = \frac{k \cdot 50}{kp_y} = \frac{50}{p_y}$$

We see that x and y remain the same even if all the demand-determinants are increased by the same multiplier, k. Hence the demand functions are homogeneous of degree zero in absolute prices and income.

❖ **Example 5.**

An individual's utility function is given by $U = x^{\alpha}.y^{\beta}$. If p_x and p_y are the fixed prices of two goods x and y and the individual's fixed income is μ, find the demand functions. Deduce that the elasticity of demand for either good with respect to income or its price is equal to unity.

❍ **Solution :**

Maximise utility function $U = x^{\alpha}y^{\beta}$ subject to budget constraint : $\mu = xp_x + yp_y$. We form the Lagrange expression

$$F = x^{\alpha}y^{\beta} + \lambda(xp_x + yp_y - \mu)$$

First order conditions for maximum utility require

$$\frac{\delta F}{\delta x} = \alpha x^{\alpha-1}.y^{\beta} + \lambda p_x = 0 \quad \text{...(1)}$$

$$\frac{\delta F}{\delta y} = \beta y^{\beta-1}.x^{\alpha} + \lambda p_y = 0 \quad \text{...(2)}$$

$$\frac{\delta F}{\delta \lambda} = xp_x + yp_y - \mu = 0 \quad \text{...(3)}$$

From (1) $\alpha x^{\alpha-1}y^{\beta} = -\lambda p_x$ or, $\lambda = \frac{-\alpha x^{\alpha-1}y^{\beta}}{p_x}$

From (2) $\lambda = \frac{-\beta y^{\beta-1}x^{\alpha}}{p_y}$. Thus we get

$$\frac{-\alpha x^{\alpha-1}y^{\beta}}{p_x} = \frac{-\beta y^{\beta-1}x^{\alpha}}{p_y} \quad \text{or,} \quad \frac{\alpha x^{\alpha-1}}{x^{\alpha}p_x} = \frac{\beta y^{\beta-1}}{y^{\beta}p_y}$$

$$\frac{\alpha x^{\alpha-1-\alpha}}{p_x} = \frac{\beta y^{\beta-1-\beta}}{p_y} \quad \text{or,} \quad \frac{\alpha}{x.p_x} = \frac{\beta}{yp_y}$$

$$\frac{\alpha y}{p_x} = \frac{\beta x}{p_y} \quad \text{or,} \quad yp_y = \left(\frac{\beta}{\alpha}\right).xp_x$$

Now the demand function for the good x is obtained by substituting

$yp_y = \left(\frac{\beta}{\alpha}\right)xp_x$ in the budget constraint.

$$xp_x + \left(\frac{\beta}{\alpha}\right)xp_x = \mu \quad \text{or,} \quad xp_x\left(1+\frac{\beta}{\alpha}\right) = \mu$$

$$\text{or,} \quad \left(\frac{xp_x(\alpha+\beta)}{\alpha}\right) = \mu \quad \text{or,} \quad x = \frac{\alpha\mu}{p_x(\alpha+\beta)}.$$

By putting $x.p_x = \left(\frac{\alpha}{\beta}\right)yp_y$ in equation (3) we get

$$\left(\frac{\alpha}{\beta}\right)yp_y + yp_y = \mu \quad \text{or,} \quad yp_y\left(\frac{\alpha}{\beta}+1\right) = \mu$$

$$\text{or,} \quad \frac{yp_y(\alpha+\beta)}{\beta} = \mu \quad \text{or,} \quad y = \frac{\beta\mu}{(\alpha+\beta)p_y}$$

Absolute value of elasticity of demand for good x w.r. to its price

$$= \frac{-p_x}{x}\cdot\frac{\delta x}{\delta p_x} = \frac{-p_x}{x}\cdot\frac{\delta}{\delta p_x}\left[\frac{\alpha\,\mu}{p_x(\alpha+\beta)}\right]$$

$$= \frac{-p_x}{x}\cdot\left[\frac{-\alpha\,\mu}{p_x^2(\alpha+\beta)}\right] = \frac{\alpha\,\mu}{xp_x(\alpha+\beta)} = \frac{1}{x}.x = 1$$

Elasticity of demand for good x w. r. to income

$$= \frac{\mu}{x}\cdot\frac{\delta x}{\delta\mu} = \frac{\mu}{x}\cdot\frac{\delta}{\delta\mu}\left[\frac{\alpha\mu}{(\alpha+\beta)p_x}\right] = \frac{\mu}{x}\left[\frac{\alpha}{(\alpha+\beta)p_x}\right] = \frac{1}{x}.x = 1.$$

❖ **Example 6.**

An individual's preference scale for the goods X and Y is defined by the MRS of Y on X : R $= \frac{(x-a)}{(y-b)}$. Show that $u = (x-a)^2 + (y-b)^2$ is one form of the utility function.

❍ **Solution :**

We know that MRS $= -\frac{dy}{dx}$. So

$$\frac{-dy}{dx} = \frac{x-a}{y-b} \quad \text{or,} \quad (y-b)dy + (x-a)dx = 0 \quad \text{or,} \quad (y-b)dy = -(x-a)dx$$

Integrating

$$\frac{y^2}{2} - by = -\left[\frac{x^2}{2} - ax\right] + c \quad \text{[where } c = \text{constant]}$$

$$\text{or,} \quad y^2 - 2by = -(x^2 - 2ax) + 2c$$

$$\text{or,} \quad (y^2 - 2by + b^2) = -(x^2 - 2ax + a^2) + (2c + b^2 + a^2)$$

$$\text{or,} \quad (y-b)^2 + (x-a)^2 = (2c + b^2 + a^2)$$

$$\text{or,} \quad (y-b)^2 + (x-a)^2 = u \text{ [where } u = 2c + b^2 + a^2\text{]}$$

This is one form of the utility function for which the MRS is equal to $\frac{(x-a)}{(y-b)}$.

❖ **Example 7.**

If $U = \log\left[(x+a)^{\alpha}.(y+b)^{\beta}\right]$ is one form of a utility function, find the marginal rate of substitution between the two goods X and Y and deduce that the elasticity of substitution is $\sigma = 1 + \dfrac{b\alpha x + a\beta y}{(\alpha+\beta)xy}$.

❍ **Solution :**

Here $U = \log(x+a)^{\alpha} + \log(y+b)^{\beta}$

or, $U = \alpha\log(x+a) + \beta\log(y+b)$

$$\therefore MU_x = \frac{\delta U}{\delta x} = \frac{\alpha}{x+a} \text{ and } MU_y = \frac{\delta U}{\delta y} = \frac{\beta}{y+b}$$

$$MRS_{xy} = \frac{MU_x}{MU_y} = \frac{\alpha}{x+a} \times \frac{y+b}{\beta} = \frac{\alpha}{\beta}\cdot\frac{y+b}{x+a}.$$

Let $r = \dfrac{MU_x}{MU_y} = \dfrac{\alpha}{\beta}\cdot\dfrac{y+b}{x+a}$.

$$\therefore r = \frac{\alpha}{\beta}\cdot\frac{y+b}{x+a} = \frac{-dy}{dx}\cdot \text{ Then } \log r = \log\left(\frac{\alpha}{\beta}\right) + \log\frac{y+b}{x+a}$$

or, $\log r = \log\alpha - \log\beta + \log(y+b) - \log(x+a)$

By total differential

$$\frac{1}{r}dr = \frac{1}{y+b}dy - \frac{1}{x+a}dx.$$

$$\text{or, } \frac{dr}{r} = \frac{(x+a)dy - (y+b)dx}{(x+a)(y+b)}$$

Putting $dy = -rdx$ we get

$$\frac{dr}{r} = \frac{(x+a)(-rdx) - (y+b)dx}{(x+a)(y+b)} = \frac{-dx\{(x+a)r + (y+b)\}}{(x+a)(y+b)}$$

$$\text{Also } d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \text{ or, } d\left(\frac{y}{x}\right) = \frac{dx(-rx-y)}{x^2} = \frac{-dx(rx+y)}{x^2}$$

$$\therefore \frac{d(y/x)}{(y/x)} = \frac{-dx(rx+y)}{x^2} \times \frac{x}{y} = \frac{-dx(rx+y)}{xy}.$$

$$\text{Now } \sigma = \frac{d(y/x)/(y/x)}{dr/r} = \frac{-dx(rx+y).(x+a)(y+b)}{xy[-dx\{(x+a)r + (y+b)\}]}$$

Putting $r = \dfrac{\alpha(y+b)}{\beta(x+a)}$ so that

$$\sigma = \frac{\left\{\dfrac{\alpha(y+b)x}{\beta(x+a)} + y\right\}(x+a)(y+b)}{xy\left[\dfrac{(x+a)\alpha(y+b)}{\beta(x+a)} + (y+b)\right]}$$

$$= \frac{\left[\dfrac{\alpha x(y+b) + \beta y(x+a)}{\beta(x+a)}\right](x+a)(y+b)}{xy\left[\dfrac{\alpha(x+a)(y+b) + \beta(x+a)(y+b)}{\beta(x+a)}\right]}$$

$$= \left[\frac{\alpha x(y+b) + \beta y(x+a)}{\beta(x+a)}\right](x+a)(y+b) \times \frac{\beta(x+a)}{xy[\alpha(x+a)(y+b) + \beta(x+a)(y+b)]}$$

$$= [\alpha x(y+b) + \beta y(x+a)](x+a)(y+b) \times \frac{1}{xy(x+a)(y+b)[\alpha+\beta]}$$

$$= \frac{\alpha x(y+b) + \beta y(x+a)}{xy(\alpha+\beta)} = \frac{\alpha xy + \alpha xb + \beta xy + \beta ay}{xy(\alpha+\beta)}$$

$$= \frac{xy(\alpha+\beta) + \alpha xb + \beta ay}{xy(\alpha+\beta)} = 1 + \frac{b\alpha x + \beta ay}{xy(\alpha+\beta)}.$$

❖ **Example 8.**

A consumer buys only two goods x and y. No other goods exist and there is no possibility of saving. The marginal utility of x is independent of the quantity of y consumed and the MU of y is independent of the quantity of x consumed. The MU of x is constant no matter how much he consumes, but the MU of y falls as consumption increases. In the initial equilibrium he consumes some of each good. Determine : (a) the slope of the IC, (b) the equation of the utility function, (c) the curvature of the IC, (d) whether the marginal utility of money is constant, rising or falling as money income increases, (e) the nature of the price elasticity of demand for x.

❍ **Solution :**

(a) Let $U = U(x, y)$ be the utility function. Now MU_x = constant = α (say), and $MU_y = \beta/y$ such that MU_y decreases as y increases where β is a constant.

Slope of IC $= \dfrac{dy}{dx} = \dfrac{-MU_x}{MU_y} = \dfrac{-\alpha}{\beta/y} = \dfrac{-\alpha y}{\beta} < 0.$

So the indifference curve is negatively sloped.

(b) Now $\dfrac{dy}{dx} = \dfrac{-\alpha}{\beta}y$. or, $\beta\dfrac{dy}{y} = -\alpha dx$

or, $\beta\displaystyle\int\frac{dy}{y} = -\alpha\int dx$ or, $\beta \log y = -\alpha x + u$ [u = constant]

or, $u = \alpha x + \beta \log y$

The utility function is $u = \alpha x + \beta \log y$.

(c) The I.C. will be convex to the origin if $\frac{d^2y}{dx^2} > 0$. Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{-\alpha}{\beta}y\right] = -\frac{\alpha}{\beta}\left[\frac{dy}{dx}\right] = \frac{-\alpha}{\beta} \times \frac{-\alpha}{\beta}y = \frac{\alpha^2}{\beta^2}\cdot y > 0$$

The I.C. is thus convex to the origin.

(d) Our problem is to maximise $u = \alpha x + \beta \log y$ subject to $M = p_x x + p_y y$. Form the Lagrange expression $L = \alpha x + \beta \log y + \lambda[M - p_x . x - p_y y]$ where λ is the Lagrange multiplier. First order conditions of maximisation then require

$$\frac{\delta L}{\delta x} = \alpha - \lambda p_x = 0 \quad \text{..........} \quad (1)$$

$$\frac{\delta L}{\delta y} = \beta / y - \lambda p_y = 0 \quad \text{...........} \quad (2)$$

$$\frac{\delta L}{\delta \lambda} = M - p_x . x - p_y . y = 0 \quad \text{............} \quad (3)$$

Here λ is the MU of money. Now from (3)

$$M = p_x x + p_y y \quad \text{or, } M = \frac{\alpha}{\lambda}x + \frac{\beta}{\lambda y}\cdot y \quad \text{or, } M = \frac{\alpha x}{\lambda} + \frac{\beta}{\lambda}$$

$$\text{or, } M = \frac{\alpha x + \beta}{\lambda} \quad \text{or, } \lambda M = \alpha x + \beta \quad \text{or, } \lambda = \frac{\lambda x + \beta}{M}.$$

As M increases λ decreases. Hence the MU of money is falling.

(e) Now in equilibrium

$$MRS_{xy} = \frac{p_x}{p_y} \quad \text{or, } \frac{\alpha y}{\beta} = \frac{p_x}{p_y} \quad \text{or, } y = \left(\frac{\beta}{\alpha}\right)\left(\frac{p_x}{p_y}\right).$$

Again $M = p_x . x + p_y . y = p_x . x + p_y \left[\frac{\beta}{\alpha}\cdot\frac{p_x}{p_y}\right]$

$$= p_x\left[x + \frac{\beta}{\alpha}\right] \quad \text{or, } x + \frac{\beta}{\alpha} = \frac{M}{p_x}.$$

Thus $x = \frac{M}{p_x} - \frac{\beta}{\alpha}$ is the demand function for x, and $\frac{\delta x}{\delta p_x} = -\frac{M}{p_x^2}$. Then

$$e_{p_x} = \frac{p_x}{x}\cdot\frac{\delta x}{\delta p_x} = \frac{p_x}{x} \times \frac{-M}{p_x^2} = \frac{-M}{xp_x} \quad \text{or, } \left|e_{p_x}\right| = \frac{M}{xp_x} = \frac{p_x \cdot x + p_y \cdot y}{x \cdot p_x} > 1.$$

Thus the price elasticity of demand is greater than unity in absolute value.

❖ Example 9.

Let the utility function and the budget constraint be given by $V = xy$ and $100 - p_x x - p_y y = 0$. (a) Find the demand functions for x and y. (b) Show that these functions are homogeneous of degree zero in absolute prices and income. **[C.U. 1991]**

○ **Solution :**

(a) To solve, we have to first maximise $V = xy$ subject to $100 - p_x.x - p_y.y = 0$. Let $W = xy + \lambda(100 - p_x.x - p_y.y)$ where λ is the Lagrange multiplier. First order conditions of maximisation require.

$$\frac{\delta W}{\delta x} = y - \lambda p_x = 0 \quad \text{.................... (1)}$$

$$\frac{\delta W}{\delta y} = x - \lambda p_y = 0 \quad \text{.................... (2)}$$

$$\frac{\delta W}{\delta \lambda} = 100 - p_x . x - p_x . y = 0 \quad \text{.................. (3)}$$

Combining (1) and (2) we get, $\frac{y}{x} = \frac{\lambda p_x}{\lambda p_y}$ or, $x \cdot p_x = y \cdot p_y$. Putting this result in (3) we get

$$2p_x . x = 100 \quad \text{or,} \quad p_x \cdot x = 50 \quad \text{or,} \quad x = \frac{50}{p_x}.$$

Thus $x = \frac{50}{p_x}$ is the demand function for x. Note that this can be written alternatively as $x = \frac{100}{2p_x}$ or, $x = \frac{M}{2p_x}$, where M = total money income = 100. Thus demand function for x is a function of M and p_x, i.e, $x = x(M. p_x)$. Similarly we get $y = \frac{50}{p_y}$ as the demand function for y and here also $y = y(M, p_y)$.

(b) we get $x = x(M, p_x) = \frac{50}{p_x}$. Multiplying each factor by λ we get

$$x(\lambda M, \lambda p_x) = \frac{50\lambda}{\lambda p_x} = \frac{50}{p_x} = \lambda^0 x(M, p_x).$$

Similarly, $$x(\lambda M, \lambda p_y) = \frac{50\lambda}{\lambda p_y} = \frac{50}{p_y} = \lambda^0 y(M, p_y)$$

This shows that the demand functions are homogeneous of degree zero.

❖ **Example 10.**

A conumer has the utility function $U = x^\alpha y^\beta$ such that $0 < \alpha < 1$ and $0 < \beta < 1$. Show that (i) there is diminishing marginal utility to increased consumption of either commodity, (ii) the indifference curves are downward sloping, (iii) MU of one commodity increases as the consumption of the other commodity increases.

○ **Solution :**

(i) $U = x^\alpha y^\beta$. Hence

$$MU_x = \frac{\delta U}{\delta x} = \alpha x^{\alpha-1} y^\beta, \quad MU_{xx} = \frac{\delta}{\delta x}\left(\frac{\delta U}{\delta x}\right) = \alpha(\alpha-1)x^{\alpha-2}y^\beta$$

Since $\alpha < 1$, $MU_{xx} < 0$. Again

$$MU_y = \frac{\delta U}{\delta y} = \beta x^{\alpha} y^{\beta-1};\; MU_{yy} = \frac{\delta}{\delta y}\left(\frac{\delta U}{\delta y}\right) = \beta(\beta-1)x^{\alpha}y^{\beta-2}$$

Since $\beta < 1$, $MU_{yy} < 0$.

This shows that there is diminishing marginal utility.

(ii) The slope of the I. C. is given by

$$-\frac{MU_x}{MU_y} = -\frac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = \frac{-\alpha}{\beta}\frac{y}{x} < 0 \quad [\alpha > 0, \beta > 0]$$

The indifference curves have negative slopes.

(iii) $MU_{xy} = \frac{\delta}{\delta y}\left(\frac{\delta U}{\delta x}\right) = \alpha x^{\alpha-1}\beta y^{\beta-1} = \alpha\beta x^{\alpha-1}y^{\beta-1} > 0.$

Marginal utility of x increases as y increases. We need no further calculation. Since $MU_{xy} = MU_{yx}$, marginal utility of y also increases as x increases.

❖ **Example 11.**

Prove that the utility maximising quantities x_1 and x_2 are the same whether we maximise $u(x_1,x_2)$ or $W = f[u(x_1,x_2)]$, where f is a strictly increasing function of u.

[C.U. 1987]

or,

Show that if a consumer maximies his utility subject to the budget constraint for one given utility index, he will behave in identical fashion irrespective of the utility index chosen, as long as the index selected is a monotonic transformation of the original one.

[C.U. 1994]

❍ **Solution :**

The objective of the consumer is to maximise $u = f(x_1, x_2)$ subject to the budget constrzaint $y_0 = p_1x_1 + p_2x_2$. We apply Lagrange multiplier method to solve this problem. Form the function $V = f(x_1, x_2) + \lambda (y_0 - p_1x_2 - p_2x_2)$ where λ is the Lagrange multiplier. The first order conditions of maximisation require :

$$\frac{\delta V}{\delta x_1} = f_1 - \lambda p_1 = 0 \qquad \text{....... (i)}$$

$$\frac{\delta V}{\delta x_2} = f_2 - \lambda p_2 = 0 \qquad \text{....... (ii)}$$

$$\frac{\delta V}{\delta \lambda} = y_0 - p_1x_1 - p_2x_2 = 0 \qquad \text{....... (iii)}$$

From (i) and (ii) we get

$$f_1 = \lambda p_1 \text{ and } f_2 = \lambda p_2 \text{ or, } f_1 / f_2 = p_1 / p_2$$

The second order condition of maximisation requires that

$$2f_{12}p_1p_2 - f_{22}p_1^2 - f_{11}p_2^2 > 0$$

The original utility function is $u = f(x_1, x_2)$. Now suppose we form a new utility index $W = F(y) = F[f(x_1, x_2)]$ by monotonic transformation of the original utility index. The function $F(u)$ is then a monotonic increasing function of u, i.e, $F'(u) > 0$. It can be demonstrated that maximizing W subject to the budget constraint is equivalent to maximising U subject to the budget constraint. We form the function $Z = F[f(x_1, x_2)] + \lambda(y_0 - p_1x_1 - p_2x_2)$. To maximise Z the first order conditions of maximisation require :

$$\frac{\delta Z}{\delta x_1} = F' f_1 - \lambda p_1 = 0 \;\; \ldots\ldots \text{(iv)}$$

$$\frac{\delta Z}{\delta x_2} = F' f_2 - \lambda p_2 = 0 \;\; \ldots\ldots \text{(v)}$$

$$\frac{\delta Z}{\delta \lambda} = y_0 - p_1x_1 - p_2x_2 = 0 \;\; \ldots\ldots \text{(vi)}$$

From (iv) and (v) we get

$$\frac{f_1}{f_2} = \frac{p_1}{p_2}$$

This proves that the first-order conditions are invariant with respect to the particular choice of the utility index. It can be seen that the second order conditions are also fulfilled. Hence it is seen that if a consumer maximises utility subject to the budget constraint for any given utility index, he will behave in the identical fashion if there is a monotonic transformation of the utility index.

❖ **Example 12.**

Show by using methods of calculus the relation between diminishing marginal rate of substitution and the convexity of indifference curves. **[C.U. 1982]**

○ **Solution :**

For the utility function $u = f(q_1, q_2)$ the equation of an I.C. can be written as $u_0 = f(q_1, q_2)$. From the total differential of this equation we get $du_0 = f_1dq_1 + f_2dq_2$, where $f_1 = \frac{\delta u}{\delta q_1}$ and $f_2 = \frac{\delta u}{\delta q_2}$. Since u_0 is constant.

$$du_0 = 0. \Rightarrow f_1dq_1 + f_2dq_2 = 0 \Rightarrow \frac{dq_2}{dq_1} = -\frac{f_1}{f_2}.$$

But $\frac{dq_2}{dq_1}$ is the slope of the I.C. From the non-satiety assumption $f_1 > 0$ and $f_2 > 0$. Hence $\frac{dq_2}{dq_1} < 0$. The negative of the slope of the I.C. at any point is called the Marginal Rate of Substitution (MRS) of q_1 for q_2.

$$\text{MRS} = -\frac{dq_2}{dq_1} = \frac{f_1}{f_2}$$

Now the MRS is diminishing if $\frac{d(\text{MRS})}{dq_1} < 0$, i.e, $\frac{d(f_1 / f_2)}{dq_1} < 0$. But $f_1 = f_1(q_1, q_2)$

so that

$$df_1 = f_{11}dq_1 + f_{12}dq_2$$

$$\therefore \frac{df_1}{dq_1} = f_{11} + f_{12}\frac{dq_2}{dq_1}.$$

Similarly $\frac{df_2}{dq_1} = f_{21} + f_{22}\frac{dq_2}{dq_1}$. Now

$$\frac{d(f_1 / f_2)}{dq_1} = \frac{f_2 \frac{df_1}{dq_1} - f_1 \frac{df_2}{dq_1}}{f_2^{\,2}} = \frac{\left(f_{11} + f_{12}\frac{dq_2}{dq_1}\right)f_2 - f_1\left(f_{21} + f_{22}\frac{dq_2}{dq_1}\right)}{f_2^{\,2}}$$

Thus $\frac{d(f_1 / f_2)}{dq_1} < 0$ implies that

$$\frac{f_{11}f_2 + f_{12}f_2\frac{dq_2}{dq_1} - f_1f_{21} - f_1f_{22}\frac{dq_2}{dq_1}}{f_2^{\,2}} < 0$$

$$\text{or,} \quad \frac{f_{11}f_2 - f_{12}f_2\frac{f_1}{f_2} - f_1f_{21} + f_1f_{22}\frac{f_1}{f_2}}{f_2^{\,2}} < 0$$

$$\text{or,} \quad \frac{f_{11}f_2^{\,2} - 2f_{12}f_1f_2 + f_{22}f_1^{\,2}}{f_2^{\,3}} < 0 \quad \text{........... (1)}$$

Hence the MRS between two commodities is diminishing. Now convexity of the I.C requires $\frac{d^2q_2}{dq_1^{\,2}} > 0$. But

$$\frac{d^2q_2}{dq_1^{\,2}} = -\frac{1}{f_2^{\,3}}(f_{11}f_2^{\,2} - 2f_{12}f_1f_2 + f_{22}f_1^{\,2})$$

Thus the I.C. will be convex when $(f_{11}f_2^{\,2} - 2f_{12}f_1f_2 + f_{22}f_1^{\,2}) < 0$ which is the same condition as (1), i.e, diminishing MRS. This is the relation between diminishing MRS and the convexity of I.C's.

❖ Example 13.

An individual lives in a two-commodity world. How will his purchases change if prices and money income are doubled ? **[C.U. 1993]**

❍ Solution :

Let the utility function of an individual be $u = f(q_1, q_2)$. The objective of the individual is to maximise $u = f(q_1, q_2)$ subject to the budget constraint $y_0 = p_1q_1 + p_2q_2$. We apply Lagrange multiplier method to solve this problem. Form the function $V = f(q_1, q_2) + \lambda(y_0 - p_1 q_2 - p_2 q_2)$ where λ is the Lagrange multiplier. First order conditions of maximisation require :

$$\frac{\delta V}{\delta q_1} = f_1 - \lambda p_1 = 0 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{\delta V}{\delta q_2} = f_2 - \lambda p_2 = 0 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

$$\frac{\delta V}{\delta \lambda} = y_0 - p_1 q_1 - p_2 q_2 = 0 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

From (i) and (ii) we get

$$f_1 = \lambda p_1 \text{ and } f_2 = \lambda p_2 \text{ i.e, } \frac{f_1}{f_2} = \frac{p_1}{p_2}$$

Let us now suppose that prices and money income are doubled. The budget constraint then becomes $2y_0 - 2p_1 q_1 - 2p_2 q_2 = 0$. The function V now becomes $v = f(q_1, q_2) + \lambda(2y_0 - 2p_1 q_1 - 2p_2 q_2)$. The first order conditions are

$$\frac{\delta v}{\delta q_1} = f_1 - 2\lambda p_1 = 0 \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$\frac{\delta v}{\delta q_2} = f_2 - 2\lambda p_2 = 0 \quad \ldots\ldots\ldots\ldots \text{(v)}$$

$$\frac{\delta v}{\delta \lambda} = 2y_0 - 2p_1 q_1 - 2p_2 q_2 = 0 \quad \ldots\ldots\ldots\ldots \text{(vi)}$$

Equation (vi) can be written as $2(y_0 - p_1 q_1 - p_2 q_2) = 0$. Since $2 \neq 0$ we get $y_0 - p_1 q_1 - p_2 q_2 = 0$. Again from equations (iv) and (v) we get

$$\frac{f_1}{f_2} = \frac{p_1}{p_2}$$

Thus we see that when prices and money income are doubled the equilibrium conditions remain the same. Hence the quantities purchased by the consumer will remain unaffected.

❖ **Example 14.**

If $R = \frac{\alpha}{\beta} \cdot \frac{y+b}{x+a}$ is the MRS of y for x, show that one form of the utility function is $u = (x + a)^{\alpha} \cdot (y + b)^{\beta}$ where a, b, α and β are given constants. **[B. U. 1999]**

❍ **Solution :**

we are given $R = \frac{\alpha}{\beta} \cdot \frac{y+b}{x+a}$

$$\text{i.e, } -\frac{dy}{dx} = \frac{\alpha(y+b)}{\beta(x+a)} \qquad \left[\text{since MRS of } y \text{ for } x = \left|\frac{dy}{dx}\right|\right]$$

i.e, $\frac{1}{\alpha(y+b)}dy = -\frac{1}{\beta(x+a)}dx$

or, $\frac{\beta dy}{(y+b)} = \frac{-\alpha\, dx}{(x+a)}$

Integrating both sides we get

$$\int \frac{\beta dy}{(y+b)} = -\int \frac{\alpha dx}{(x+a)}$$

or, $\beta \log (y + b) + \log c_1 = -\alpha \log (x + a) + \log c_2$

or, $\log (y + b)^\beta + \log (x + a)^\alpha = \log c_2 - \log c_1$

or, $\log \{(y + b)^\beta . (x + a)^\alpha \} = \log (c_2/c_1) = \log u$ (say)

$\therefore\ u = (x + a)^\alpha, (y + b)^\beta$

❖ **Example 15.**

A consumer has a utility function $U(x_1, x_2) = (x_1 - a_1)^\alpha (x_2 - a_2)^{1-\alpha}$, where $0 < \alpha < 1$ and $a_1 > 0, a_2 > 0$. (a) Find the demand fuctions for two goods. (b) How would you interpret a_1 and a_2?

❍ **Solution :**

Let p_1 be the price per unit of x_1, p_2 be the price per unit of x_2 and M be the money income of the consumer. Then our problem is to maximise $U = (x_1 - a_1)^\alpha (x_2 - a_2)^{1-\alpha}$ subject to $M = p_1x_1 + p_2x_2$. Let us form the expression

$L = (x_1 - a_1)^\alpha (x_2 - a_2)^{1-\alpha} + \lambda (M - p_1x_1 - p_2x_2)$

where λ is the Lagrange multipler. The first order conditions of maximisation require

$$\frac{\delta L}{\delta x_1} = \alpha (x_1 - a_1)^{\alpha-1} (x_2 - a_2)^{1-\alpha} - \lambda p_1 = 0 \quad \text{... (1)}$$

$$\frac{\delta L}{\delta x_2} = (1 - \alpha)(x_2 - a_2)^{-\alpha} (x_1 - a_1)^\alpha - \lambda p_2 = 0 \quad \text{... (2)}$$

$$\frac{\delta L}{\delta \lambda} = M - p_1x_1 - p_2x_2 = 0 \quad \text{... (3)}$$

From (1) and (2) we get

$$\frac{\alpha(x_1 - a_1)^{\alpha-1}(x_2 - a_2)^{1-\alpha}}{(1-\alpha)(x_1 - a_1)^\alpha (x_2 - a_2)^{-\alpha}} = \frac{\lambda p_1}{\lambda p_2}$$

or, $\frac{\alpha(x_2 - a_2)}{(1-\alpha)(x_1 - a_1)} = \frac{p_1}{p_2}$

or, $\alpha p_2 (x_2 - a_2) = (1 - \alpha)p_1 (x_1 - a_1)$

or, $p_2 (x_2 - a_2) = \frac{1-\alpha}{\alpha} p_1(x_1 - a_1)$

or, $p_2x_2 = \frac{1-\alpha}{\alpha} p_1 (x_1 - a_1) + p_2a_2$

Putting this value in (3) we get

$$M - p_1x_1 - \frac{1-\alpha}{\alpha} p_1(x_1 - a_1) - p_2a_2 = 0$$

or, $p_1x_1 + \frac{1-\alpha}{\alpha} p_1(x_1 - a_1) = M - p_2a_2$

or, $p_1x_1 + \frac{1-\alpha}{\alpha} p_1x_1 = M - p_2a_2 + \frac{1-\alpha}{\alpha} p_1a_1$

or, $x_1\left(p_1 + \frac{1-\alpha}{\alpha} p_1\right) = M - p_2a_2 + \frac{a_1p_1(1-\alpha)}{\alpha}$

or, $x_1\left(\frac{p_1\alpha + p_1 - p_1\alpha}{\alpha}\right) = M - p_2a_2 + \frac{a_1p_1}{\alpha} - a_1p_1$

$\therefore \quad x_1 = \frac{\alpha}{p_1}\left(M - p_2a_2 + \frac{a_1p_1}{\alpha} - a_1p_1\right)$

$\therefore \quad x_1 = a_1 + \frac{\alpha}{p_1}(M - a_1p_1 - a_2p_2)$

This is the required demand function for x_1. Similarly the demand function for x_2 will be given by $x_2 = a_2 + \frac{1-\alpha}{p_2}(M - a_1p_1 - a_2p_2)$.

(b) The parameters a_1 and a_2 can be regarded as minimum subsistence quantities of the two commodities x_1 and x_2 respectively. The utility function will have the domain $x_1 > a_1$ and $x_2 > a_2$. Thus positive utility will be obtained only if the consumption exceeds subsistence requirements.

❖ Example 16.

Assume a demand function $q = f(p) = 25 - 5p$

(a) What is the quantity demanded if the price is Rs. 3? (b) Assume the demand is 18 units. What is the corresponding price ? (c) What would be the demand if the commodity in question were a free good ? (d) What is the highest price anybody will pay for the commodity ?

❍ Solution :

(a) If price is Rs. 3 per unit the quantity demanded is obtained from the demand function by putting $p = 3$. Thus $q = f(3) = 25 - 5(3) = 10$ units.

(b) When demand is 18 units, $q = 18$. Then,

$$18 = 25 - 5p \text{ or, } 5p = 7 \text{ or, } p = 1.40$$

(c) If the commodity is a free good, then $p = 0$. When $p = 0$,

$$q = 25 - 5(0) = 25$$

(d) When $q = 0$, $25 - 5p = 0$ or, $p = 5$. Thus the highest price is Rs. 5. Actually when price is Rs. 5 demand is zero. Hence price must be less than Rs. 5 in order that any of the commodity be sold in the market.

❖ Example 17.

Assume the demand function $p + q = 1$. (a) Find the total revenue function. (b) Find the price if $\frac{1}{2}$ unit is sold. (c) Find the total revenue if $\frac{1}{3}$ unit is sold. (d) Find the marginal revenue function.

❍ Solution :

(a) From the demand function we get, $p = 1 - q$. Let R represent total revenue. Then $R = p.\,q = (1 - q)q$ or, $R = q - q^2$ is the total revenue function.

(b) when $q = \frac{1}{2}$, $p = 1 - \frac{1}{2} = \frac{1}{2}$

(c) when $q = \frac{1}{3}$, $R = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$

(d) Marginal revenue is given by $\frac{dR}{dq}$. Now differentiating the total revenue function we get $\frac{dR}{dq} = 1 - 2q$

❖ Example 18.

Given the demand function $p = 1 - q$, (a) find the expression for the price elasticity of demand. (b) Find the value of elasticity when $q = \frac{1}{4}$. (c) Verify the result that, MR $=$ price $\left(1 - \frac{1}{|e|}\right)$

❍ Solution :

(a) When $p = 1 - q$ we get $\frac{dp}{dq} = -1$. Hence $\frac{dq}{dp} = -1$.

Now $e = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{1-q}{q}(-1) = -\frac{1-q}{q}$

(b) When $q = \frac{1}{4}$, $e = \dfrac{1-\frac{1}{4}}{\frac{1}{4}} = -3$

(c) Total revenue (R) $= pq = (1 - q)q = q - q^2$. Then MR $=$ Marginal revenue $= \frac{dR}{dq} = 1 - 2q$. Now we know

$$\text{Price}\left(1 - \frac{1}{|e|}\right) = (1 - q)\left(1 - \frac{1}{\frac{1-q}{q}}\right) = (1 - q)\left(1 - \frac{q}{1-q}\right)$$

$$= (1 - q)\left(\frac{1-q-q}{1-q}\right) = 1 - 2q$$

Thus we get MR $=$ price $\left(1 - \frac{1}{|e|}\right)$

❖ **Example 19.**

For the demand function $q = 30 - 4p - p^2$ (i) Find the elasticity of demand when $p = 3$; (ii) Find the marginal revenue when $p = 3$.

○ **Solution :**

(i) When $q = 30 - 4p - p^2$, $\frac{dq}{dp} = -4 - 2p$. When $p = 3$

$\frac{dq}{dp} = -4 - 2(3) = -10$. Again, when $p = 3$, $q = 30 - 4(3) - (3)^2 = 9$

Now elasticity of demand (e) $= \frac{p}{q} \cdot \frac{dq}{dp} = \frac{3}{9}(-10) = -\frac{30}{9} = -3.3.$

(ii) Here $|e| = 3.3$. Now from the relation between AR, MR and elasticity of demand we get

$$MR = AR\left(1 - \frac{1}{|e|}\right) = 3\left(1 - \frac{1}{3.3}\right) = 2.1$$

❖ **Example 20.**

Let the demand function for a commodity X be $q_x = 25 - 2p_x + p_y$, where p_x and p_y are prices of X and Y respectively. Find the own price elasticity of demand and the cross price elasticity of demand for X when $p_x = 3$ and $p_y = 1$

○ **Solution :**

From the demand function $q_x = 25 - 2p_x + p_y$ we get,

$$\frac{\delta q_x}{\delta p_x} = -2 \quad \text{and} \quad \frac{\delta q_x}{\delta p_y} = 1.$$

Now own price elasticity of demand for X

$$= \frac{p_x}{q_x} \cdot \frac{\delta q_x}{\delta p_x} = \frac{p_x}{25 - 2p_x + p_y} \cdot (-2) = \frac{-2p_x}{25 - 2p_x + p_y}$$

When $p_x = 3$ and $p_y = 1$, own price elasticity of demand for X

$$= \frac{-2 \cdot (3)}{25 - 2.(3) + 1} = \frac{-6}{20} = -0.3$$

Cross price elasticity of demand for X

$$= \frac{p_y}{q_x} \cdot \frac{\delta q_x}{\delta p_y} = \frac{p_y}{25 - 2p_x + p_y}$$

When $p_x = 3$ and $p_y = 1$, cross price elasticity of demand for

$$X = \frac{1}{25 - 2(3) + 1} = \frac{1}{20} = 0.05$$

❖ **Example 21.**

Consider a demand function $q = Ap^{\alpha}y^{\beta}$, where q represents quantity demanded, p represents price, y represents income, A, α and β are constants. Determine the direct price elasticity and income elasticity of demand.

❍ **Solution :**

When $q = Ap^{\alpha}y^{\beta}$

$$\frac{\delta q}{\delta p} = A\alpha \cdot p^{\alpha-1}y^{\beta} \text{ and } \frac{\delta q}{\delta y} = A \cdot \beta p^{\alpha} y^{\beta-1}$$

Now direct price elasticity of demand $= \frac{p}{q} \cdot \frac{\delta q}{\delta p} = \frac{p}{Ap^{\alpha} y^{\beta}} \cdot A\alpha p^{\alpha-1} y^{\beta} = \alpha$

Income elasticity of demand $= \frac{y}{q} \cdot \frac{\delta q}{\delta y} = \frac{y}{Ap^{\alpha} \cdot y^{\beta}} \cdot A.\beta p^{\alpha} \cdot y^{\beta-1} = \beta.$

❖ **Example 22.**

Suppose the consumers will demand 40 units of a product when the price is Rs. 12 per unit and 25 units when the price is Rs. 18 each. Find the demand function assuming that it is linear. Also determine the TR, AR and MR functions.

❍ **Solution :**

Let the demand function be : $p = a + bx$ (since it is assumed to be linear), where p is the price per unit and x is the quantity demanded at this price. Since $x = 40$, when $p = 12$ and $x = 25$ when $p = 18$ we get

$$12 = a + 40b \quad \text{... (1)}$$
$$18 = a + 25b \quad \text{... (2)}$$

Solving (1) and (2), we get $a = 28$ and $b = \frac{2}{5}$. Hence the demand function is $p = 28 - \frac{2}{5}x$.

The total revenue function is : $R = px = \left(28 - \frac{2}{5}x\right) \cdot x = 28x - \frac{2}{5}x^2$

The marginal revenue function is : $MR = \frac{dR}{dx} = 28 - \frac{4}{5}x$

❖ **Example 23.**

Show that the demand curve $p = \left(\frac{a}{x+b}\right) - c$ is downward sloping and convex from below. Do the same properties hold for the MR curve ? **[D.U.1985]**

❍ **Solution :**

Differentiating $p = \left(\frac{a}{x+b}\right) - c$ with respect to x, we have

$$\frac{dp}{dx} = \frac{-a}{(x+b)^2} < 0 \qquad [\because a > 0 \text{ and } (x+b)^2 > 0]$$

Thus the demand curve is downward sloping. Differentiating again we have

$$\frac{d^2p}{dx^2} = \frac{2a}{(x+b)^3} > 0 \quad [a, x, b \text{ are positive}]$$

This implies that the demand curve is convex from below. Hence the demand curve is downward sloping and convex from below.

Now $R(x) = p.\, x = \dfrac{ax}{x+b} - cx$

$$MR(x) = \frac{dR(x)}{dx} = \frac{(x+b)a - ax}{(x+b)^2} - c = \frac{ab}{(x+b)^2} - c$$

Since $\dfrac{dMR(x)}{dx} = \dfrac{-2ab}{(x+b)^3} < 0$ the MR curve is downward sloping

Again, $\dfrac{d^2(MR)}{dx^2} = \dfrac{6ab}{(x+b)^4} > 0$. which implies the MR curve is convex from below.

Thus the same property holds for the MR curve also.

❖ **Example 24.**

It is given that a demand curve is convex from below, i.e, $\left(\dfrac{d^2p}{dx^2}\right) > 0$ at all points, where p is the price and x is the quantity demanded. Show that the marginal revenue curve is also convex from below if either $\dfrac{d^3p}{dx^3}$ is positive or $\dfrac{d^3p}{dx^3}$ is negative and numerically less than $\dfrac{3}{x}\cdot\dfrac{d^2p}{dx^2}$. Does a similar property hold for demand curves that are concave from below ?

[D.U. 1981, 1984]

❍ **Solution :**

Let the demand function be $p = f(x)$. Then the corresponding total revenue function is given by : $R(x) = px$

$$\frac{dR(x)}{dx} = MR(x) = P + x \cdot \frac{dp}{dx}$$

Differrentiating MR w. r. to x we get

$$\frac{dMR(x)}{dx} = \frac{dp}{dx} + \left[x \cdot \frac{d^2p}{dx^2} + \frac{dp}{dx}\right] = \frac{2dp}{dx} + x \cdot \frac{d^2p}{dx^2}$$

Differentiating again, we have $\dfrac{d^2MR(x)}{dx^2}$ must be positive, i.e,

$$3\,\frac{d^2p}{dx^2} + x \cdot \frac{d^3p}{dx^3} > 0 \text{ or, } \frac{3}{x}\cdot\frac{d^2p}{dx^2} + \frac{d^3p}{dx^3} > 0 \quad [\text{Dividing both sides by } x\,]$$

If $\frac{d^3p}{dx^3}$ is positive, then $\frac{3}{x}\frac{d^2p}{dx^2}+\frac{d^3p}{dx^3}$ will be positive as $\frac{d^2p}{dx^2}$ is already positive. So corresponding MR (x) is convex from below.

If $\frac{d^3p}{dx^3}<0$ and $\frac{3}{x}\frac{d^2p}{dx^2}>\frac{d^3p}{dx^3}$, then again $\frac{d^2\text{MR}(x)}{dx^2}>0$. Thus MR is convex also if $\frac{d^3p}{dx^3}<0$ but less than $\frac{3}{x}\frac{d^2p}{dx^2}$.

In case the demand curve is concave from below, $\frac{d^2p}{dx^2}$ will be negative. Then (i) $\frac{d^2\text{MR}(x)}{dx^2}=\frac{3}{x}\frac{d^2p}{dx^2}+\frac{d^3p}{dx^3}$, will be positive if $\frac{d^3p}{dx^3}>0$ and $\frac{3}{x}\frac{d^2p}{dx^2}<\frac{d^3p}{dx^3}$ which is not fulfilled. Thus MR (x) will not be convex from below if the demand curve is concave from below. MR (x) will be concave from below (ii) If $\frac{d^3p}{dx^3}<0$, then $\frac{3}{x}\frac{d^2p}{dx^2}+\frac{d^3p}{dx^3}<0$

So MR (x) is again concave from below.

❖ **Example 25.**

A manufacturer determines that t employees will produce a total of x units of a product per day where $x = 5t$. If the demand equation for the product is $p = -0.5x + 40$, determine the marginal revenue product when $t = 2$. Interpret your result.

❍ **Solution :**

If R is the total revnue, then $R = px = (-0.5x + 40)x = -0.5x^2 + 40x = = -0.5(5t)^2 + 40(5t)$ [$\because x = 5t$] $= -12.5t^2 + 200t$. Thus

Marginal revenue product $= \frac{dR}{dt} = -25t + 200$.

When t = 2, $\frac{dR}{dt} = (-25)(2) + 200 = 150$. This means that if a third employee is hired, the extra revenue generated is approximately 150.

❖ **Example 26.**

If $\in$ is the elasticity of $f(x)$, then show that the elasticities of $x\,f(x)$ and $f(x)/x$ are $(\in + 1)$ and $(\in - 1)$, respectively. Check with $f(x) = ax^{\alpha}$. **[D. U. 1990]**

❍ **Solution :**

(i) The elasticity of the function $x f(x)$

$$= \frac{x}{xf(x)}\frac{d}{dx}[xf(x)] = \frac{1}{f(x)}\left[x\frac{d}{dx}f(x)+1.f(x)\right]$$

$$= \frac{x}{f(x)}\cdot\frac{d}{dx}[f(x)]+1 = \in + 1.$$

Similarly the elasticity of the function $\frac{f(x)}{x}$

$$= \frac{x}{\frac{f(x)}{x}} \frac{d}{dx}\left\{\frac{f(x)}{x}\right\}$$

$$= \frac{x^2}{f(x)}\left[\frac{x\frac{d}{dx}f(x) - f(x)}{x^2}\right]$$

$$= \frac{x^2}{f(x)}\left[\frac{x\frac{d}{dx}f(x)}{x^2} - \frac{f(x)}{x^2}\right]$$

$$= \frac{x}{f(x)} \cdot \frac{d}{dx}[f(x)] - 1 = \in - 1$$

(ii) Elasticity of the function ax^α

$$= \frac{x}{ax^\alpha}\frac{d}{dx}\left[ax^\alpha\right] = \frac{x}{ax^\alpha}\alpha a x^{\alpha-1} = ax^{\alpha-1+1-\alpha} = \alpha$$

Elasticity of the function $x.ax^\alpha$ or. $ax^{\alpha+1}$

$$= \frac{x}{ax^{\alpha+1}} \cdot \frac{d}{dx}\left[ax^{\alpha+1}\right] = \frac{xa(\alpha+1).x^\alpha}{ax^{\alpha+1}} = (\alpha+1).x^{\alpha+1-\alpha-1} = (\alpha+1)$$

Elasticity of the function ax^α/x or, $ax^{\alpha-1}$

$$= \frac{x}{ax^{\alpha-1}} \cdot \frac{d}{dx}\left(ax^{\alpha-1}\right)$$

$$= \frac{x.(\alpha-1)ax^{\alpha-2}}{ax^{\alpha-1}} = (\alpha-1)x^{\alpha-2+1-\alpha+1} = (\alpha-1)$$

Thus the results are verified.

❖ Example 27.

Suppose that p is the price per box of biscuits when x boxes are demanded, and $x = 75 - p^2$. (a) Find the price elasticity of demand when the price of a box of biscuits is Rs 7.50. (b) What percent decrease in the price would yield about a 5 percent increase in the demand ?

❍ Solution :

(a) The given demand function is : $x = 75 - p^2$. Differentiating w. r. to p, we get $\frac{dx}{dp} = -2p$. Hence

$$|\eta_d| = \frac{-pdx}{x\,dp} = \frac{-p}{(75-p^2)}.(-2p) = \frac{2p^2}{75-p^2}$$

When $p = 7.50, |\eta_d| = \frac{2.(7.50)^2}{75-(7.50)} = 6$

(b) From the result of part (a), it can be said that a decrease of 1% in the unit price would cause an approximate increase of 6% in the demand. Hence demand would increase by 5% when the decrease in price is (5/6) or 0.83%.

❖ **Example 28.**

A company can sell 4500 items of a product when the price is Rs 5. When the price of an item is increased by 10%, the demand drops of 4250 items per month. Assuming that the demand equation is linear, (i) find the point elasticity of demand at the new price level, (ii) approximate the change in demand if the price is increased by additional 10%.

❍ **Solution :**

Suppose the linear demand function is of the form

$$x = a + bp \quad \text{....................(1)}$$

We know that $x = 4500$ when $p =$ Rs 5, and $x = 4250$ when $p = 5.50$. Substituting these values into (1) produces a pair of equations $4500 = a + 5b$ and $4250 = a + 5.5b$. Solving them we get $a = 7000$ and $b = -500$. Thus the demand functions is : $x = 7000 - 500p$.

$$\text{Now } \eta = \frac{p}{f(p)} \times f'(p) = \frac{p \times (-500)}{7000 - 500p} = \frac{-p}{14 - p}$$

(i) When $p = 5.5,\ \eta = \frac{-5.5}{14 - 5.5} = -0.65$

(ii) At a price level of Rs 5.50, a 10% increase in price will result in a percentage change in demand of approximately $\eta \times 10\% = -0.65 \times 10\% = -6.5\%$. Thus the demand will decrease by 6.5%.

❖ **Example 29.**

If the demand law is $x = 20/(p + 1)$, find elasticity of demand at the point when $p = 3$. What type of curve is the demand function ? What is its economic importance ?

❍ **Solution :**

(a) The given demand function is : $x = \frac{20}{p+1} = 20(p+1)^{-1}$. Differentiating w. r. to p we get $\frac{dx}{dp} = -20(p+1)^{-2}$. Hence

$$|\eta_d| = \frac{-p}{x} \cdot \frac{dx}{dp} = -\{p/20(p+1)^{-1}\} \times \{-20(p+1)^{-2}\}$$

$$= \frac{p(p+1)}{20} \times \frac{20}{(p+1)^2} = \frac{p}{(p+1)}$$

At $p = 3$, $|\eta_d| = \frac{3}{3+1} = \frac{3}{4}$. The demand function is a rectangular hyperbola. It is downward sloping and convex to the origin. As p increases $|\eta_d|$ approaches unity.

❖ Example 30.

Find the elasticity of demand for each of the demand laws : (a) $p=\sqrt{a-bx}$, (b) $p = x.ex^{-2}$, (c) $p = x^a e^{-b(x+c)}$ where a, b, c are constants. Also find the values of x when elasticity of demand equals unity in each case.

❍ Solution :

(a) Differentiating the demand law $p = (a - bx)^{1/2}$ w. r. t. p we have

$$1=\frac{1}{2}(a-bx)^{-1/2}(-b)\frac{dx}{dp} \quad \text{or, } 1=-\frac{b}{2(a-bx)^{1/2}}\cdot\frac{dx}{dp}$$

or, $\frac{-2}{b}(a-bx)^{1/2}=\frac{dx}{dp}$.

$$\therefore |\eta_d| = \frac{-p}{x}\frac{dx}{dp} = \frac{-(a-bx)^{1/2}}{x}\left\{\frac{2(a-bx)^{1/2}}{(-b)}\right\} = \frac{2(a-bx)}{bx}$$

Now $|\eta_d| = 1$ when $\frac{2(a-bx)}{bx} = 1$ or, $2a - 2bx = bx$.

or, $2a = 3bx$ or, $x = 2a/3b$.

(b) $p = x.ex^{-2}$ or, $\log p = \log x + x^{-2}\log e$ or, $\log p = \log x + x^{-2}$

Differentiating both sides w.r.t. p

$$\frac{1}{p}=\frac{1}{x}\cdot\frac{dx}{dp}-2x^{-3}\cdot\frac{dx}{dp} \quad \text{or, } \frac{1}{p}=\frac{dx}{dp}\left(\frac{1}{x}-\frac{2}{x^3}\right)$$

Multiplying both sides with p/x

$$\therefore \frac{p}{x}\frac{dx}{dp}\left(\frac{1}{x}-\frac{2}{x^3}\right)=\frac{1}{p}\cdot\frac{p}{x} \quad \text{or, } \frac{p}{x}\frac{dx}{dp}=\frac{1}{x}\times\frac{x^3}{x^2-2}$$

$$\therefore \eta_d=\frac{x^2}{x^2-2} \qquad \therefore |\eta_d|=\frac{x^2}{2-x^2}$$

Now $|\eta_d| = 1$ if $x^2 = 2 - x^2$ or, $2x^2 = 2$ or, $x^2 = 1$ or, $x = 1$.

(c) $p = x^a e^{-b(x+c)}$ or, $\log p = \log x^a + \log e^{-b(x+c)}$

or, $\log p = a\log x - b(x + c)\log e$ or, $\log p = a\log x - bx - bc$ [$\because \log e = 1$]

Differentiating both sides w. r. to p

$$\frac{1}{p}=a.\frac{1}{x}\frac{dx}{dp}-b.\frac{dx}{dp} \quad \text{or, } \frac{1}{p}=\left(\frac{a}{x}-b\right)\cdot\frac{dx}{dp} \quad \text{or, } \frac{dx}{dp}\left(\frac{a}{x}-b\right)=\frac{1}{p}$$

$$\text{or, } \frac{p}{x}\cdot\frac{dx}{dp}\left(\frac{a-xb}{x}\right)=\frac{1}{p}\cdot\frac{p}{x} \quad \text{or, } \frac{p}{x}\cdot\frac{dx}{dp}=\frac{1}{x}\cdot\frac{x}{a-xb}$$

$$\therefore \eta_d=\frac{1}{a-xb}. \quad \text{Now } |\eta_d|=\frac{1}{bx-a}$$

$\therefore |\eta_d| = 1$, when $bx - a = 1$ or, $bx = 1 + a$ or, $x = \frac{1+a}{b}$.

❖ **Example 31.**

Verify $|\eta_d| = \dfrac{AR}{AR - MR}$ for the linear demand law : $p = a + bx$. **[D. U. 1989]**

○ **Solution :**

Differentiating the demand law $p = a + bx$ w. r. to p, we get

$$1 = b.\frac{dx}{dp}. \quad \text{or} \quad \frac{1}{b} = \frac{dx}{dp}$$

$$\therefore \quad |\eta_d| = \frac{-p}{x}.\frac{dx}{dp} = -\left(\frac{a+bx}{x}\right).\left(\frac{1}{b}\right) = -\left(\frac{a+bx}{bx}\right) \text{......(1)}$$

Again, $R(x) = p.x = (a + bx).x = ax + bx^2$. Then

$$AR = \left(\frac{R}{x}\right) = a + bx \text{ and } MR = R'(x) = a + 2bx$$

$$\text{So } \frac{AR}{AR - MR} = \frac{a+bx}{(a+bx)-(a+2bx)} = \frac{a+bx}{-bx} = \frac{-(a+bx)}{bx} \text{ ......(2)}$$

From (1) and (2) we conclude $|\eta_d| = \dfrac{AR}{AR - MR}$.

❖ **Example 32.**

Verify the relationship $MR = p\left[1 - \dfrac{1}{|\eta_d|}\right]$ for the demand function : $p = (12 - x)^{1/2}$ when $0 \le x \le 12$.

○ **Solution :**

Differentiating $p = (12 - x)^{1/2}$ w. r. to p, we get

$$1 = \frac{1}{2}(12 - x)^{-1/2}(-1).\frac{dx}{dp} \qquad \text{or, } 1 = \frac{-1}{2(12-x)^{1/2}}.\frac{dx}{dp}$$

$$\text{or, } \frac{dx}{dp} = -2.(12 - x)^{1/2}.$$

$$\therefore \quad |\eta_d| = \frac{-p}{x}.\frac{dx}{dp} = \frac{-(12-x)^{1/2}}{x}.\left[-2(12-x)^{1/2}\right] = \frac{2(12-x)}{x}$$

Now $R(x) = p.x = (12 - x)^{1/2}x$. Then

$$MR = R'(x) = \frac{1}{2}(12 - x)^{-1/2}.(-1)x + (12 - x)^{1/2}$$

$$= \frac{-x}{2(12-x)^{1/2}} + (12-x)^{1/2}$$

$$= \sqrt{12-x} - \frac{x}{2\sqrt{12-x}} = \frac{2(12-x)-x}{2\sqrt{12-x}} = \frac{24-3x}{2\sqrt{12-x}} \text{ .... (1)}$$

Also, $p\left(1-\frac{1}{|\eta_d|}\right) = (12-x)^{1/2}\left\{1-\frac{x}{2(12-x)}\right\}$

$$= (12-x)^{1/2}\left[\frac{24-3x}{2(12-x)}\right] = \frac{24-3x}{2\sqrt{12-x}} \qquad \text{..... (2)}$$

From (1) and (2) we conclude MR = $p\{1-(1/|\eta_d|)\}$

❖ Example 33.

Show that for maximum total revenue, elasticity of demand is unity.

○ Solution :

Let $x = f(p)$ be the demand curve. Then $R = p.f(p) = p.x$.

$$\therefore \text{MR} = \frac{dR}{dx} = \frac{d}{dx}(p.x) = x.\frac{dp}{dx} + p.1.$$

Since for R to be maximum dR/dx must be zero we get,

$$\frac{dR}{dx} = 0 \text{ or, } x\frac{dp}{dx} + p = 0 \text{ or, } x = -p(dx/dp)$$

$$\therefore |\eta_d| = \frac{-p}{x}.\frac{dx}{dp} = \frac{-p}{-p.(dx/dp)} \times (dx/dp) = 1$$

This shows that $|\eta_d|$ must be unity if total revenue is to be maximised.

❖ Example 34.

For the demand function $aQ + bP - k = 0$, where a, b and k are positive constants, determine the point elasticity of demand when MR is zero. **[D. U. 1994]**

○ Solution :

Here $P = \frac{-aQ}{b} + \frac{k}{b}$. Then

$$R = PQ = \frac{-aQ^2}{b} + \frac{kQ}{b} \text{ and } MR = \frac{dR}{dQ} = \frac{-2aQ}{b} + \frac{k}{b}.$$

Setting MR = 0 we get,

$$\frac{-2aQ}{b} + \frac{k}{b} = 0 \text{ or, } -2aQ + k = 0 \text{ or, } Q = \frac{k}{2a}.$$

Now $\frac{dP}{dQ} = \frac{-a}{b}$. When $Q = \frac{k}{2a}$, $P = \frac{-ak}{2ab} + \frac{k}{b} = \frac{-k+2k}{2b} = \frac{k}{2b}$.

Now $|\eta_d| = \frac{-dQ}{dP}.\frac{P}{Q} = -\left[\frac{-b}{a}.\frac{k/2b}{k/2a}\right] = \left[\frac{b}{a}\frac{k \times 2a}{2b \times k}\right] = 1.$

❖ Example 35.

The MR function is given by $R'(x) = 28 - 15x + 2x^2$. Find the TR function and the demand function.

❍ **Solution :**

Let R (x) be total revenue function, so that $\frac{dR}{dx} = 28 - 15x + 2x^2$. Thus

$$dR = (28 - 15x + 2x^2)dx$$

Integrating both sides we get

$$R(x) = \int (28 - 15x + 2x^2)dx = 28x - \frac{15x^2}{2} + \frac{2x^3}{3} + K.$$

Since it is obvious that R $(x) = 0$, when $x = 0$, we get $K = 0$.

$$\therefore R(x) = 28x - \left(\frac{15}{2}\right)x^2 + \left(\frac{2}{3}\right)x^3.$$

If $f(x)$ is the demand function, $R(x) = x.f(x)$. So

$$f(x) = \frac{R(x)}{x} = 28 - \left(\frac{15}{2}\right)x + \left(\frac{2}{3}\right)x^2.$$

If we let p rupees be the price of one unit of the commodity when x units are demanded, then because $p = f(x)$, the demand equation is

$$p = 28 - \left(\frac{15}{2}\right)x + \left(\frac{2}{3}\right)x^2.$$

❖ **Example 36.**

If MR function for output x is given by $MR = \frac{6}{(x+2)^2} + 5$, find the TR function and the demand function.

❍ **Solution :**

Here $MR = \frac{dR}{dx} = \frac{6}{(x+2)^2} + 5$. Hence, $dR = \left\{\frac{6}{(x+2)^2} + 5\right\}dx$

Integrating both sides we get

$$R(x) = \int \{6.(x+2)^{-2} + 5\}dx = -6(x+2)^{-1} + 5x + K \qquad \text{(K is a constant).}$$

Since it is obvious that $R(x) = 0$ when $x = 0$ we get,

$$0 = -6(0 + 2)^{-1} + 5 \times 0 + K \quad \text{or, } K = 3.$$

$$\text{Then TR} = 5x - \frac{6}{(x+2)} + 3 = 5x + 3 - \frac{6}{(x+2)} = 5x + \frac{3x+6-6}{x+2} = 5x + \frac{3x}{x+2}$$

If $f(x)$ is the demand function, $R(x) = x.f(x)$. So $f(x) = 5 + \left(\frac{3}{x+2}\right)$. If we let '$p$' rupees be the price of one unit of the commodity when x units are demanded, then because $p = f(x)$, the demand function is

$$p = 5 + \left(\frac{3}{x+2}\right)$$

❖ **Example 37.**

If the marginal revenue function of a firm is $MR = \frac{ab}{(x+b)^2} - c$, find TR and show that $p = \frac{a}{x+b} - c$ is the demand function.

○ **Solution :**

$MR = \frac{dR}{dx} = ab(x+b)^{-2} - c$. So $dR = \{ab(x+b)^{-2} - c\}dx$.

Integrating both sides we get

$$R(x) = \int \{ab(x+b)^{-2} - c\}dx = ab\frac{(x+b)^{-1}}{-1} - cx + K$$

Since it is obvious that R(0) = 0, when $x = 0$, we get

$$0 = \frac{-ab}{b} + K \quad \text{or, } K = a$$

$$\therefore TR = \frac{-ab}{(x+b)} - cx + a = \left(a - \frac{ab}{x+b}\right) - cx$$

$$= \left[\frac{ax+ab-ab}{x+b} - cx\right] = \frac{ax}{x+b} - cx = \left[\frac{a}{x+b} - c\right]x$$

If $f(x)$ is the demand function, $R(x) = x.f(x)$, so that $f(x) = \left[\frac{a}{(x+b)} - c\right]$. If we let p rupees be the price of one unit of the commodity when x units are demanded then because $p = f(x)$, the demand equation $p = \left[\frac{a}{(x+b)} - c\right]$ is the required demand law.

❖ **Example 38.**

Derive the demand function for which the elasticity of demand is the constant k.

○ **Solution :**

Let p be the price and x be the quantity demanded. Then, by definition, elasticity of demand η_d is given by

$$\eta_d = \frac{p}{x}\frac{dx}{dp} = k \quad \text{or, } \frac{dx}{x} = k.\frac{dp}{p}$$

Integrating $\int \frac{dx}{x} = k\int \frac{dp}{p}$

or, $\log x = k \log p + \log \lambda$ [λ = constant]

or, $\log x = \log p^k + \log \lambda$ or, $\log x = \log (p^k.\lambda)$

Here $x = p^k.\lambda$ is the required demand function.

❖ **Example 39.**

Obtain the demand function for a commodity whose elasticity of demand is given as $\eta = a - bp$.

❍ **Solution :**

$$\eta = \frac{p}{x}\frac{dx}{dp} = a - bp \text{ or, } \left[\frac{a-bp}{p}\right]dp = \frac{dx}{x} \text{ or, } \left[\frac{a-bp}{p}\right]\cdot dp - \frac{dx}{x} = 0.$$

Integrating $(a \log p - bp) - \log x = \log \lambda$ [λ = constant]

or, $\log p^a + \log e^{-bp} - \log x = \log \lambda$ [$\because \log e = 1$]

or, $\log\left(\frac{p^a.e^{-bp}}{x}\right) = \log\lambda$ or, $\frac{p^a.e^{-bp}}{x} = \lambda$ or, $x = \frac{p^a.e^{-bp}}{\lambda}$.

This is the required demand function.

❖ **Example 40.**

Determine the equation of a curve which has price elasticity of demand as unity throughout its range.

❍ **Solution :**

Let p be the price and x be the quantity demanded. Then

$|\eta_d| = \frac{-p}{x}\frac{dx}{dp} = 1$ or, $\frac{dx}{x} = \frac{-dp}{p}$ or, $\int\frac{dx}{x} = -\int\frac{dp}{p}$

or, $\log x = -\log p + \log c$ [c = constant]

or, $\log x = \log p^{-1} + \log c$ or, $\log x = \log\left(\frac{c}{p}\right)$

$\therefore x = \frac{c}{p}$ or, $px = c$. Hence $px = c$ is the required demand function.

❖ **Example 41.**

If y is the number of persons having income of Rs x, it is found that y decreases as x increases according to the law $\frac{dy}{dx} = -\frac{my}{x}$, where m is a given constant. Show that the dependence of y on x is $y = ax^{-m}$.

❍ **Solution :**

Here $\frac{dy}{dx} = -m\cdot\frac{y}{x}$ or, $\frac{dy}{y} = -m\frac{dx}{x}$

Integrating

$\int\frac{dy}{y} = -m\int\frac{dx}{x}$ or, $\log y = -m\log x + \log a$ [where a = constant]

or, $\log y = \log x^{-m} + \log a$ or, $\log y = \log(ax^{-m})$ or, $y = ax - m$.

❖ **Example 42.**

The following are the two demand functions for two commodities X_1 and X_2 :

$$X_1 = p_1^{-1.7} . p_2^{0.8}, \qquad X_2 = p_1^{0.5} . p_2^{-0.8}$$

Determine whether the two commodities are complements or substitutes.

○ **Solution :**

Given $\quad X_1 = p_1^{-1.7} . p_2^{0.8}$

Then $\quad \frac{\delta X_1}{\delta p_2} = p_1^{-1.7} . 0.8 p_2^{0.8-1} = 0.8 p_1^{-1.7} p_2^{-0.2}$

Again since $\quad X_2 = p_1^{0.5} p_2^{-0.8}$

$$\frac{\delta X_2}{\delta p_1} = 0.5 p_1^{0.5-1} . p_2^{-0.8} = 0.5 p_1^{-0.5} p_2^{-0.8}$$

Here $\frac{\delta X_1}{\delta p_2} > 0$ and $\frac{\delta X_2}{\delta p_1} > 0$. Hence the two goods are substitutes of each other.

❖ **Example 43.**

The demand function for a commodity is given by

$$X_1 = 300 - 0.5\, p_1^2 + 0.02 p_2 + 0.05 y.$$

Find the income elasticity of demand when $p_1 = 12$, $p_2 = 10$, and $y = 200$.

○ **Solution :**

Since $X_1 = 300 - 0.5 p_1^2 + 0.02\, p_2 + 0.05 y$, we get $\frac{\delta X_1}{\delta y} = 0.05$

Income elasticity of demand is given as

$$\eta = \frac{y}{X_1} \frac{\delta X_1}{\delta y} = \frac{y}{300 - 0.5 p_1^2 + 0.02 p_2 + 0.05 y} . \frac{\delta X_1}{\delta y}$$

$$= \frac{200}{300 - 0.5 \times 144 + 0.02 \times 10 + 0.05 \times 200} \times 0.05$$

$$= \frac{200}{300 - 72 + 0 \cdot 2 + 10} \times 0.05$$

$$= \frac{200 \times 0.05}{238.2} = \frac{10}{238.2} = 0.04.$$

The commodity is a normal necessity.

❖ **Example 44.**

Obtain the income elasticity of demand for X_1 when the demand function is

$$X_1 = \frac{2}{p_1} - 4 p_2 + 5 y^2 \quad \text{and} \quad p_1 = 2,\ p_2 = 0.25,\ y = 100.$$

❍ **Solution :**

$$X_1 = \frac{2}{p_1} - 4p_2 + 5y^2 \text{. Hence } \frac{\delta X_1}{\delta y} = 10y.$$

$$\eta = \frac{y}{\frac{2}{p_1} - 4p_2 + 5y^2} \cdot \frac{\delta X_1}{\delta y} = \frac{100}{\frac{2}{2} - 4 \times .0.25 + 5(100)^2} \times 10 \times 100 = 2$$

The commodity is a normal luxury.

❖ **Example 45.**

The TR is given by R = R(q). Now if R(q = 2) = 200 and MR = $10 + 20q - 3q^2$ find the TR function of the firm.

❍ **Solution :**

Here $\frac{dR}{dq} = 10 + 20q - 3q^2$ or, $dR = (10 + 20q - 3q^2).\ dq$

Integrating both sides we get

$$\int dR = \int \left(10 + 20q - 3q^2\right) dq$$

$$\text{or,} \quad R(q) = 10q + \frac{20q^2}{2} - \frac{3q^3}{3} + K \text{ where K is a constant.}$$

or, $R(q) = 10q + 10q^2 - q^3 + K$

Now it is given that R = 200 when q = 2. Thus

$200 = 20 + 40 - 8 + K$ or, $200 - 52 = K$ or, $148 = K$

∴ $R(q) = 10q + 10q^2 - q^3 + 148$. It is the required TR function of the firm.

❖ **Example. 46.**

Consider a consumer in a two commodity world whose indifierence map is such that the slope of I. C. is everywhere equal to $-y/x$. (a) Show that the demand for X is independent of the price of y and the price elasticity for X is unitary. (b) Find the value of the equilibrium marginal rate of substitution given that the prices of x and y are Re 1 and Rs 2 per unit and income is Rs 120. (c) What does the Engel curve for x look like ? What is the income elasticity of demand for x?

❍ **Solution :**

(a) Slope of the indifference curve $= \frac{dy}{dx} = -y/x$. Let $M = p_x x + p_y y$ be the budget equation of the consumer. Now the slope of budget line, $\frac{dy}{dx} = -\frac{p_x}{p_y}$. In equilibrium $\frac{y}{x} = \frac{p_x}{p_y}$. or, $yp_y = xp_y$. Let us put $xp_x = yp_y$ in the budget equation.

Then

$$M = p_x x + p_y y = yp_y + y.p_y = 2yp_y \quad \text{or,} \quad y = \frac{M}{2p_y}$$

This is the demand function for y. Similarly $x = \frac{M}{2p_x}$ is the demand function for x.

(b) In equilibrium $MRS_{xy} = \frac{p_x}{p_y}$. Since $p_x = 1$, $p_y = 2$, we get $MRS_{xy} = \frac{1}{2}$

(c) Let $x = \frac{M}{2p_x}$ be the demand function for x, i.e, $x = f(M, p_x)$. Now income elasticity of demand for x is given by

$$e_M = \frac{\% \text{ change in } x}{\% \text{ change in M}} = \frac{\delta x / x}{\delta M / M} = \frac{\delta x}{x} \times \frac{M}{\delta M} = \frac{M}{x}\frac{\delta x}{\delta M} = \frac{\delta \log x}{\delta \log M}$$

Let $x = \frac{M}{2p_x}$. Then $\log x = \log M - \log(2p_x)$ or, $\log x = \log M - \log 2 - \log p_x$

$$\text{Then } e_M = \frac{\delta \log x}{\delta \log M} = 1.$$

Hence the income elasticity of demand for x is one. Again

$$e_{p_x} = \frac{\delta \log x}{\delta \log p_x} = -1$$

The Engel curve for x is a straight line through the origin.

❖ Example 47.

A consumer's demand curve for x is given by $P = 100 - \sqrt{Q}$. Calculate his point-price elasticity of demand when the price of x is 60.

○ Solution :

$$P = 100 - \sqrt{Q} \quad \text{or,} \quad \sqrt{Q} = 100 - P \quad \text{or,} \quad Q = (100 - P)^2.$$

Then $$\frac{dQ}{dP} = -2(100 - P)$$

$$E_P = \frac{dQ/Q}{dP/P} = \frac{P}{Q}\frac{dQ}{dP} = \frac{P}{(100-P)^2} \times -2(100 - P) = \frac{-2P}{(100-P)}.$$

When $P = 60$, $E_p = \frac{-120}{40} = -3$.

❖ Example 48.

A consumer is observed to purchase $q_1 = 20$, $q_2 = 10$ at prices $p_1 = 2$, $p_2 = 6$. She is also observed to purchase $q_1 = 18$, $q_2 = 4$ at prices $p_1 = 3$, $p_2 = 5$. Is her behaviour consistent with the axioms of the Theory of Revealed Preference ?

[B. U. 2001]

○ Solution :

Let $X = \begin{Bmatrix} q_1 & q_2 \\ 20 & 10 \end{Bmatrix}$ and $Y = \begin{Bmatrix} q_1 & q_2 \\ 18 & 4 \end{Bmatrix}$

Case 1. Cost of the 1st bundle (X) at prices $p_1 = 2, p_2 = 6$, is given by

$$p_1q_1 + p_2q_2 = 2 \times 20 + 6 \times 10 = \text{Rs. } 100.$$

At the same price cost of the 2nd bundle (Y) is

$$p_1q_1 + p_2q_2 = 2 \times 18 + 6 \times 4 = 36 + 24 = \text{Rs. } 60.$$

Now at prices $p_1 = 2, p_2 = 6$, when the consumer purchases X, Y is also available within the attainble set of the consumer, i.e, X is revealed preferred to Y.

Case 2. In another price income situation when $p_1 = 3$, and $p_2 = 5$, the cost of the 1st bundle is $3 \times 20 + 5 \times 10 =$ Rs. 110. At the same prices, the cost of the 2nd boundle is $3 \times 18 + 5 \times 4 = 54 + 20 =$ Rs. 74. Now at prices $p_1 = 3, p_2 = 5$ when the consumer purchases the 2nd bundle, the 1st bundle is not within the attainable set of the consumer and hence it is consistent with the weak axiom of revealed preference.

❖ **Example 49.**

Construct ordinary and compensated demand functions for the utility function $u = 2q_1q_2 + q_2$. **[C.U. 1993, B.U. 2001]**

❍ **Solution :**

Let the budget constraint be $y = p_1q_1 + p_2q_2$. The utility function is given as $u = 2q_1q_2 + q_2$. We can now form the Lagrange expression as

$$V = 2q_1q_2 + q_2 + \lambda(y - p_1q_1 - p_2q_2)$$

The first order conditions for maximum utility are :

$$\frac{\delta V}{\delta q_1} = 2q_2 - \lambda p_1 = 0 \text{ .... (i)}$$

$$\frac{\delta V}{\delta q_2} = 2q_1 + 1 - \lambda p_2 = 0 \text{ .... (ii)}$$

$$\frac{\delta V}{\delta \lambda} = y - p_1q_1 - p_2q_2 = 0 \text{ .... (iii)}$$

From (i) we have $\frac{2q_2}{p_1} = \lambda$. From (ii) we have $\frac{2q_1 + 1}{p_2} = \lambda$. Thus

$$\frac{2q_2}{p_1} = \frac{2q_1 + 1}{p_2} \quad \text{or, } 2p_2q_2 = 2p_1q_1 + p_1 \quad \text{or, } p_2q_2 = \frac{2p_1q_1 + p_1}{2}$$

From (iii) we have

$$y = p_1q_1 + p_2q_2 \qquad \text{or, } y = p_1q_1 + \frac{2p_1q_1 + p_1}{2}$$

$$\text{or, } y = \frac{2p_1q_1 + 2p_1q_1 + p_1}{2} \qquad \text{or, } 2y = 4p_1q_1 + p_1$$

$$\text{or, } \frac{2y - p_1}{4p_1} = q_1 \qquad \text{or, } q_1 = \frac{y}{2p_1} - \frac{1}{4}.$$

This is the equation for the ordinary demand function for q_1. Now

$$y = p_1\left[\frac{2y-p_1}{4p_1}\right]+p_2q_2 \quad \text{or,} \quad 4y=(2y-p_1)+4p_2q_2$$

$$\text{or,} \quad \frac{4y-2y+p_1}{4p_2}=q_2 \qquad \text{or,} \quad \frac{2y+p_1}{4p_2}=q_2 \quad \text{or,} \quad q_2=\frac{y}{2p_2}+\frac{p_1}{4p_2}.$$

This is the required equation for the ordinary demand function for q_2.

In the case of compensated demand curve our problem is to minimise $p_1q_1+p_2q_2$ subject to $u_0=2q_1q_2+q_2$. To solve this problem let us form the Lagrange expression

$$Z=p_1q_1+p_2q_2+\mu[u_0-2q_1q_2-q_2]$$

The first order conditions of minimisation require

$$\frac{\delta Z}{\delta q_1}=p_1-2\mu q_2=0 \quad \text{............ (iv)}$$

$$\frac{\delta Z}{\delta q_2}=p_2-2\mu q_1-\mu=0 \quad \text{............ (v)}$$

$$\frac{\delta Z}{\delta \mu}=u_0-2q_1q_2-q_2=0 \quad \text{.............. (vi)}$$

From (iv) we have $\frac{p_1}{2q_2}=\mu$ and from (v) we get $\frac{p_2}{2q_1+1}=\mu$. Then

$$\frac{p_1}{2q_2}=\frac{p_2}{2q_1+1}$$

$$\text{or,} \quad 2p_1q_1+p_1=2p_2q_2 \quad \text{.............. (vii)}$$

$$\text{or,} \quad \frac{2p_1q_1+p_1}{2p_2}=q_2$$

Substituting this value of q_2 in (vi) we get

$$u_0=2q_1\left[\frac{2p_1q_1+p_1}{2p_2}\right]+\frac{2p_1q_1+p_1}{2p_2}$$

$$\text{or,} \quad u_0=\frac{4p_1q_1^2+2p_1q_1+2p_1q_1+p_1}{2p_2}$$

$$\text{or,} \quad u_0=\frac{4p_1q_1^2+4p_1q_1+p_1}{2p_2}$$

$$\text{or,} \quad 2u_0p_2=4p_1q_1^2+4p_1q_1+p_1$$

$$\text{or,} \quad 4p_1q_1^2+4p_1q_1+(p_1-2u_0p_2)=0$$

This is a quadratic equation in q_1. Solving for q_1 we get

$$q_1 = \frac{-4p_1 \pm \sqrt{16p_1^2 - 4(4p_1)(p_1 - 2u_0p_2)}}{2\cdot(4p_1)}$$

Since q_1 can never be negative we get

$$q_1 = \frac{-4p_1 \pm \sqrt{16p_1^2 - 4(4p_1)(p_1 - 2u_0p_2)}}{2\cdot(4p_1)}$$

$$\text{or,} \quad q_1 = \frac{-4p_1 + \sqrt{32u_0p_1p_2}}{8p_1}$$

$$\text{or,} \quad q_1 = \frac{-p_1 + \sqrt{2u_0p_1p_2}}{8p_1}$$

This is the required equation for compensated demand function for q_1. Again from (vii) we get

$$2p_1q_1 + p_1 = 2p_2q_2 \qquad \text{or,} \quad q_1 = \frac{2p_2q_2 - p_1}{2p_1}$$

Substituting this value of q_1 in (vi) we get

$$u_0 = 2\left[\frac{2p_2q_2 - p_1}{2p_1}\right]q_2 + q_2 \qquad \text{or,} \quad u_0 = \frac{4p_2q_2^2 - 2p_1q_2 + 2p_1q_2}{2p_1}$$

$$\text{or,} \quad u_0 = \frac{4p_2q_2^2}{2p_1} \qquad \text{or,} \quad q_2 = \sqrt{\frac{u_0p_1}{2p_2}}$$

This is the required compensated demand function for q_2.

❖ **Example 50.**

If P C.C. for y is parallel to y axis what will be the shape of the demand curve for y? **[C.U. 1987]**

❍ **Solution :**

Let us consider the budget constraint as $P_x.x + P_y.y = M$. Now taking total differential we get

$$x.dP_x + P_x\,dx + P_y\,dy + y.dP_y = dM$$

Since we are concerned with the PCC for y, we are to consider here $dP_x = dM = 0$. So we get $P_x\,dx + P_y.dy + y.dP_y = 0$. Again since P.C.C. for y is parallel to y axis, $dx = 0$. Then

$$P_y.dy + y.dP_y = 0$$

$$\text{or,} \quad \frac{dy}{y} = -\frac{dP_y}{P_y}$$

Integrating we get,

$$\log y = -\log P_y + \log K \quad \text{or,} \quad \log y + \log P_y = \log K$$

$$\text{or,} \quad \log (y \cdot P_y) = \log K \quad \text{or,} \quad y.P_y = K = \text{constant.}$$

This shows that the demand curve for Y is a rectangular hyperbola.

❖ Example 51.

If P.C.C. of x is parallel to x axis then derive the shape of the demand curve for x.

○ Solution :

Let $M = p_x x + p_y y$ be the budget constraint. Then

$$d\,M = p_x.dx + x.dp_x + p_y.dy + y.dp_y \quad \ldots\ldots\ldots\ldots (1)$$

Here since M is constant, $dM = 0$. Again P.C.C. of x is parallel to x axis, i.e, $dy = 0$. Moreover when we like to derive the demand curve for x we assume price of y to be constant, i.e., $dp_y = 0$. Then equation (1) finally becomes

$$p_x dx + x dp_x = 0. \quad \text{or,} \quad p_x.dx = -x dp_x \quad \text{or,} \quad \frac{dx}{x} = \frac{-dp_x}{p_x}$$

Integrating both sides

$$\int \frac{dx}{x} = -\int \frac{dp_x}{p_x} \quad \text{or,} \quad \log x = -\log p_x + \log K$$

$$\text{or,} \quad \log x + \log p_x = \log K \quad \text{or,} \quad \log(x.p_x) = \log K$$

$$\text{or,} \quad xp_x = K = \text{constant.}$$

This shows that the demand curve for x is a rectangular hyperbola.

❖ Example 52.

Find the value of price elasticity of demand when demand function is of the following form $q = \frac{k}{p}$, where k is a constant. **[B. U. 1999]**

○ Solution :

we know, $e_d = \frac{p}{q} \cdot \frac{dq}{dp}$

Now when $q = \frac{k}{p}$, then $\frac{dq}{dp} = -\frac{k}{p^2}$

Hence, $e_d = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{\frac{k}{p}} \cdot \left(-\frac{k}{p^2}\right) = \frac{p^2}{k}\left(-\frac{k}{p^2}\right) = -1.$

Hence price elasticity of demand is -1 at all points on the given demand function. The demand function is a rectangular hyperbola.

❖ **Example 53.**

Obtain the demand function of a consumer with utility function $u = x^a y^{1-a}$, $0 < a < 1$ and show that the budget share of each commodity is constant.

[B. U. 1999]

❍ **Solution :**

Let the prices of two goods be p_x and p_y respectively and the income of the consumer be M. Now the demand function can be derived from the equilibrium condition of the consumer which states

$$\frac{f_x}{f_y} = \frac{p_x}{p_y}$$

Now, $f_x = ax^{a-1}y^{1-a} = a\left(\frac{x}{y}\right)^{a-1}$, $\quad f_y = (1-a)x^a y^{-a} = (1-a)\left(\frac{x}{y}\right)^a$

Then $$\frac{f_x}{f_y} = \frac{a\left(\frac{x}{y}\right)^{a-1}}{(1-a)\left(\frac{x}{y}\right)^a} = \frac{a}{1-a}\left(\frac{x}{y}\right)^{a-1-a} = \frac{a}{1-a}\frac{y}{x}$$

Now in equilibrium

$$\frac{a}{1-a}\cdot\frac{y}{x} = \frac{p_x}{p_y}$$

$$\text{or, } y = \frac{1-a}{a}\frac{xp_x}{p_y} \quad \text{... (i)}$$

Putting this value of y in the budget constraint we get

$$M = x\,p_x + \frac{1-a}{a}\cdot\frac{xp_x}{p_y}\cdot p_y$$

$$\text{or, } M = xp_x + \frac{1-a}{a}xp_x$$

$$\text{or, } M = xp_x\left(1 + \frac{1-a}{a}\right)$$

$$\text{or, } x = \frac{aM}{p_x} \quad \text{... (ii)}$$

This is the required demand function for X. Again putting this value of x in (i) we get

$$y = \frac{1-a}{a}\cdot\frac{aM}{p_x}\cdot\frac{p_x}{p_y}$$

$$\text{or, } y = \frac{(1-a)M}{p_y} \quad \text{... (iii)}$$

This is the required demand function of y. To find out the budget share of each commodity we have to get the values of $p_x \frac{x}{M}$ and $p_y \frac{y}{M}$. From (ii) and (iii) we get

$$p_x \frac{x}{M} = a$$

$$p_y \frac{y}{M} = (1 - a)$$

Since a is constant, it is clear that the budget share of each commodity is constant.

❖ **Example 54.**

If $q = f(p)$ and elasticity of q with respect to p is equal to α (constant), determine the demand equation. **[B.U. 1999]**

❍ **Solution :**

we are given

$$|e_d| = -\frac{p}{q}\cdot\frac{dq}{dp} = \alpha, \text{ i.e, } \frac{p}{q}\cdot\frac{dq}{dp} = -\alpha$$

$$\text{i.e, } \frac{dq}{q} = -\frac{\alpha}{p}\cdot dp$$

Integrating both sides we get

$$\int \frac{1}{q}\cdot dq = -\alpha \int \frac{1}{p}\cdot dp$$

or, $\log q + \log c_1 = -\alpha \log P + \log c_2$

or, $\log q + \log p^{\alpha} = (\log c_2 - \log c_1)$

or, $\log (q.p^{\alpha}) = \log \left(\frac{c_2}{c_1}\right)$

or, $qp^{\alpha} = c$ where $c = \frac{c_2}{c_1}$

or, $q = \frac{c}{p\alpha}$.

It is the required demand equation.

❖ **Example 55.**

Show that for the price equation $\log p = 100 - \frac{1}{2} \log q$ where q is the quantity demanded, the commodity is supposed to be a luxury good. **[B.U. 2002]**

❍ **Solution :**

For a luxury good, we know, the absolute value of price elasticity of demand will be greater than unity. Now,

$$\log p = 100 - \frac{1}{2} \log q$$

$$\text{or, } \log q = 200 - 2 \log p$$

Differentiating both sides with respect to log p we get

$$\frac{d\log q}{d\log p} = -2.$$

$$\text{But } \frac{d\log q}{d\log p} = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\cdot\frac{dq}{dp}.$$

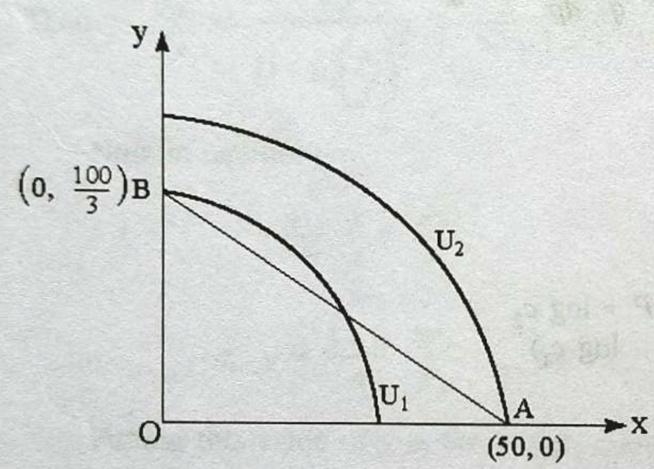
Thus $e_d = -2$, i.e. $|e_d| = 2$. Since $|e_d| > 1$, it is a luxury good.

❖ Example 56.

Suppose a consumer in a two good world has utility function $u = x^2 + y^2$. Let his budget line be given by $2x + 3y = 100$. Find out the equilibrium purchase plan and comment on your answer. **[B.U. 2002]**

❍ Solution :

If the utility function is $u = x^2 + y^2$ and if u is constant at a level u°, then the equation of the corresponding indifference curve becomes $x^2 + y^2 = u°$ which is the equation of a circle with the centre at (0, 0). Thus the indifference curves will be concave and there will be corner solution. The corner point of the feasible set at which utility is maximum will give the equilibrium purchases of the consumer. The equation of the budget line is



$$2x + 3y = 100 \text{ i.e. } \frac{2x}{100} + \frac{3y}{100} = 1$$

$$\text{or, } \frac{x}{50} + \frac{y}{\frac{100}{3}} = 1.$$

Hence the coordinates of the intersection points of the budget line with the two axe are A (50, 0) and B $\left(0, \frac{100}{3}\right)$. Now utility at A (50,0) is $(50)^2 + 0^2 = 2500$ and utility at B $\left(0, \frac{100}{3}\right)$ is $0^2 + \left(\frac{100}{3}\right)^2 = \frac{10000}{9} = 1111\frac{1}{9}$.

Since $2500 > 1111\frac{1}{9}$, utility is higher at A than at B. The indifference curve passing through A will be higher than the indifference curve passing through B. Hence A will be the equilibrium point where the consumer will purchase 50 units of X and nothing of Y.